

Solutions to Homework Set 7

1) **Critical mass:** We expand

$$n(x, t) = \sum_{m=1}^{\infty} a_m(t) \sin\left(\frac{m\pi x}{L}\right),$$

and also

$$\mu = \sum_{m, \text{odd}} \frac{4\mu}{m\pi} \sin\left(\frac{m\pi x}{L}\right), \quad 0 < x < L.$$

Substituting in the given equation, and using the linear independence of the sine functions, then gives

$$\dot{a}_m(t) = \left(\lambda - \frac{Dm^2\pi^2}{L^2}\right) a_m(t) + \frac{4\mu}{m\pi},$$

where the last term is present only when m is odd. Let us define

$$\alpha_m = \left(\lambda - \frac{Dm^2\pi^2}{L^2}\right).$$

The solution to the evolution equation is either

$$a_m(t) = \left(a_m(0) + \frac{4\mu}{m\pi\alpha_m}\right) e^{\alpha_m t} - \frac{4\mu}{m\pi\alpha_m},$$

or

$$a_m(t) = a_m(0)e^{\alpha_m t},$$

depending on whether m is odd or even. The $m = 1$ mode is the first to go unstable, and this happens as soon as $\alpha_1 > 0$, *i.e.* when $L > L_{\text{crit}}$ where

$$L_{\text{crit}} = \pi\sqrt{\frac{D}{\lambda}}.$$

To find the equilibrium distribution we solve

$$D\frac{d^2n}{dx^2} + \lambda n + \mu = 0,$$

with the boundary conditions $n(0) = n(L) = 0$. This is an inhomogeneous ODE with constant coefficients. It is therefore most easily solved by combining a complementary function

$$n_{\text{CF}}(x) = A \cos\sqrt{\lambda/D}(x - L/2),$$

which has been chosen to be symmetric about the midpoint of the slab, with the particular integral

$$n_{\text{PI}}(x) = -\frac{\mu}{\lambda}.$$

To satisfy the boundary conditions we take

$$n_{\text{equilibrium}}(x) = \frac{\mu}{\lambda} \left(\frac{\cos \sqrt{\lambda/D}(x - L/2)}{\cos \sqrt{\lambda/D}(L/2)} - 1 \right).$$

Note that this blows up (quite literally!) when $\cos \sqrt{\lambda/D}(L/2) = 0$ or when

$$\frac{\pi}{2} = \sqrt{\frac{\lambda}{D}} \frac{L}{2},$$

which is another way of determining that $L_{\text{crit}} = \pi \sqrt{D/\lambda}$.

2) Semi-infinite rod: We solve this problem by an image trick. If we compute the temperature evolution under

$$\frac{\partial \theta}{\partial t} = \frac{\partial^2 \theta}{\partial x^2}$$

of a rod that is infinite in *both* directions, but with initial data $\theta(x, 0) = -1$ for $x < 0$ and $\theta(x, 0) = 1$ for $x > 0$, then symmetry guarantees that $\theta(0, t) = 0$ for all later times. The solution to this modified problem can be written down by using the heat kernel:

$$\begin{aligned} \theta(x, t) &= \sqrt{\frac{1}{4\pi t}} \left(\int_0^\infty e^{-(x-\xi)^2/4t} d\xi - \int_{-\infty}^0 e^{-(x-\xi)^2/4t} d\xi \right) \\ &= \sqrt{\frac{1}{4\pi t}} \int_0^\infty \left(e^{(x-\xi)^2/4t} - e^{-(x+\xi)^2/4t} \right) d\xi \\ &= \sqrt{\frac{1}{4\pi t}} \left(\int_{-x}^\infty e^{-\zeta^2/4t} d\zeta - \int_x^\infty e^{-\zeta^2/4t} d\zeta \right) \\ &= \sqrt{\frac{1}{4\pi t}} \left(\int_{-x}^0 e^{-\zeta^2/4t} d\zeta - \int_x^0 e^{-\zeta^2/4t} d\zeta \right) \\ &= \frac{2}{\sqrt{4\pi t}} \int_0^x e^{-\zeta^2/4t} d\zeta \\ &= \frac{2}{\sqrt{\pi}} \int_0^{x/\sqrt{4t}} e^{-\zeta^2} d\zeta \\ &= \text{erf} \left(\frac{x}{2\sqrt{t}} \right). \end{aligned} \tag{1}$$

The solution to the homework problem is therefore

$$\theta(x, t) = \theta_0 \text{erf} \left(\frac{x}{2\sqrt{Dt}} \right).$$

We used this solution when we discussed Duhamel's method

If we follow the instructions for part b) we find that the Fourier integral solution is

$$\theta(x, t) = \lim_{\epsilon \rightarrow 0} \left\{ \frac{2}{\pi} \int_0^\infty a(k, t) \sin kx dk \right\},$$

where

$$a(k, t) = a(k, 0)e^{-k^2 t},$$

and

$$a(k, 0) = \int_0^\infty e^{-\epsilon x} \sin kx \, dx = \frac{1}{k + i\epsilon}.$$

The integral over k has no actual singularity at $k = 0$, so the $i\epsilon$ can be put safely to zero, whence

$$\theta(x, t) = I(x) = \frac{2}{\pi} \int_0^\infty \frac{\sin kx}{k} e^{-k^2 t} \, dk.$$

We now need to evaluate this integral. We observe that I is manifestly equal to zero at $x = 0$, and that

$$\frac{dI}{dx} = \frac{2}{\pi} \int_0^\infty \cos kx e^{-k^2 t} \, dk = \frac{1}{\pi} \int_{-\infty}^\infty e^{ikx} e^{-k^2 t} \, dk = \sqrt{\frac{1}{\pi t}} \exp\left\{-\frac{1}{4t} x^2\right\}.$$

Integrating up, $I(x) = \int_0^x (dI/d\xi) \, d\xi$, we therefore find that

$$\theta(x, t) = \sqrt{\frac{1}{\pi t}} \int_0^x \exp\left\{-\frac{1}{4t} \xi^2\right\} \, d\xi = \sqrt{\frac{2}{\pi}} \int_0^{x/\sqrt{4t}} \exp\{-\zeta^2\} \, d\zeta = \operatorname{erf}\left(\frac{x}{2\sqrt{t}}\right).$$

3) Jacobi's Imaginary Transformation: This formula is both an illustration of the Poisson summation formula and a key identity in the theory of theta functions. It is the basis of many *duality transformations* in statistical field theory, and is an ingredient in establishing the modular invariance of genus-zero string amplitudes.

- a) We regard the circle as the entire real line quotiented by the identification $\theta \sim \theta + 2\pi n$, $n \in \mathbf{Z}$. A delta-function blob of heat at θ' is therefore copied to image blobs at $\theta' + 2\pi n$, and the heat can make its way to θ from all of these. The heat kernel is therefore

$$G(\theta, \theta'; t) = \sqrt{\frac{1}{4\pi t}} \sum_{n=-\infty}^{\infty} \exp\left\{-\frac{1}{4t}(\theta - \theta' + 2\pi n)^2\right\}.$$

Observe that this is a periodic function of θ , as it must be.

- b) We solve the same problem – the evolution of an initial heat blob at $\theta = \theta'$ by expanding in a complete set of orthogonal functions $e^{in\theta}$ on the unit circle:

$$\phi(\theta, t) = \sum_{n=-\infty}^{\infty} a_n(t) e^{in\theta}.$$

The heat equation now tells us that

$$\phi(\theta, t) = \sum_{n=-\infty}^{\infty} a_n(0) \exp\{-tn^2 + in\theta\}.$$

and from $\phi(\theta, 0) = \delta(\theta - \theta')$ and the orthogonality properties of the functions $e^{in\theta}$ we find

$$a_n(0) = \frac{1}{2\pi} \int_0^{2\pi} \delta(\theta - \theta') e^{-in\theta} \, d\theta = \frac{1}{2\pi} e^{-in\theta'}.$$

Thus

$$G(\theta, \theta'; t) = \phi(\theta, t) = \frac{1}{2\pi} \sum_{n=-\infty}^{\infty} \exp \left\{ -tn^2 + in(\theta - \theta') \right\}$$

Equating our two expressions for $G(\theta, \theta'; t)$ and cosmetically setting $\theta' = 0$, $2t = \tau$ gives

$$\sqrt{\frac{1}{2\pi\tau}} \sum_{n=-\infty}^{\infty} \exp \left\{ -\frac{1}{2\tau}(\theta + 2\pi n)^2 \right\} = \frac{1}{2\pi} \sum_{n=-\infty}^{\infty} \exp \left\{ -\frac{1}{2}\tau n^2 + in\theta \right\}.$$

4) Seasonal Heat Waves: An E&M analogue of this problem was on the Qual Exam a couple of years ago.

As suggested, we write θ as the real part of $\theta(z)e^{in\omega t}$. Plug into the heat equation and find that

$$in\omega\theta(z) - \kappa\theta''(z) = 0.$$

The two linearly independent solutions to this ODE are

$$\theta(z) = \exp \left\{ \pm \sqrt{\frac{in\omega}{\kappa}} z \right\} = \exp \left\{ \pm \sqrt{\frac{n\omega}{2\kappa}} (1+i)z \right\}.$$

We must chose the minus sign, so that the effects of the heat fluctuations die away as the depth z becomes large. Thus

$$\theta(z) = \text{Re} \left[\exp \left\{ -\sqrt{\frac{n\omega}{2\kappa}} (1+i)z + in\omega t \right\} \right]$$

Matching these solutions to the boundary conditions at $z = 0$, we find

$$\theta(z) = \theta_0 + \sum_{n=1}^{\infty} \theta_n \exp \left\{ -\sqrt{\frac{n\omega}{2\kappa}} z \right\} \cos \left(\sqrt{\frac{n\omega}{2\kappa}} z - n\omega t \right).$$

These are damped waves propagating downwards into the ground.