1) The algebra of angular momentum – optional:

a) By using the definition $\mathbf{L} = \mathbf{R}\times\mathbf{P}$, and the canonical commutation relations $[R_i, P_j] = i\hbar\delta_{ij}$, establish the following results:

a.i) $[L_i, L_j] = i\hbar\epsilon_{ijk}L_k$;
a.ii) $[\mathbf{L}\cdot\mathbf{L}, L_i] = 0$;
a.iii) $[\mathbf{R}, \mathbf{a}\cdot\mathbf{L}] = i\hbar\mathbf{a}\times\mathbf{R}$ for c-numbers $\mathbf{a}$;
a.iv) $[R_i, L_j] = i\hbar\epsilon_{ijk}R_k$;
a.v) $[P_i, L_j] = i\hbar\epsilon_{ijk}P_k$;
a.vi) $[\mathbf{R}\cdot\mathbf{R}, L_i] = 0$;
a.vii) $[\mathbf{R}\cdot\mathbf{R}\, P_i, L_j] = i\hbar\epsilon_{ijk}\mathbf{R}\cdot\mathbf{R}\, P_k$.

b) The raising and lowering operators, $L_\pm$, are defined via $L_\pm = L_x \pm iL_y$. Use them to establish the following results:

b.i) $[L_\pm, L^2] = 0$, where $L^2 \equiv \mathbf{L}\cdot\mathbf{L}$;
b.ii) $[L_z, L_\pm] = \pm\hbar L_\pm$;
b.iii) $[L_+, L_-] = 2\hbar L_z$;
b.iv) $L^2 = L_+L_- + L_z^2 - \hbar L_z$.

c) For the Hilbert space of functions on a sphere, discuss why the set $\{L^2, L_z\}$ forms a complete set of commuting observables (CSCO), *i.e.*, argue that inclusion of L_x or L_y violates the commuting property. Would $\{L^2, L_x\}$ also form a CSCO?

d) Show that the eigenvalues of L^2 are positive or zero. Can they always be written in the form $\hbar^2 l(l+1)$ with l dimensionless and greater than or equal to zero?

e) Denote the set of simultaneous eigenstates of L^2 and L_z by $|l, m\rangle$, where

$$\begin{aligned} L^2|l,m\rangle &= \hbar^2 l(l+1)|l,m\rangle; \\ L_z|l,m\rangle &= \hbar m|l,m\rangle. \end{aligned}$$

As yet, there is no restriction on l and m, except that $l \geq 0$. We shall now derive constraints on l and m by using only the algebraic properties of the angular momentum operators, as specified by the commutation relations.

e.i) By using the hermitean property, $(L_-)^\dagger = L_+$, show that $\langle \ell, m|L_+L_-|l, m\rangle \geq 0$.
e.ii) By using part (b-iv), show that $l(l+1) - m(m-1) \geq 0$.
e.iii) Similarly, by considering $\langle \ell, m|L_-L_+|l, m\rangle$, show that $l(l+1) - m(m+1) \geq 0$.
e.iv) Hence, show that $-l \leq m \leq l$.

f) Show that the ket $L_-|l, m\rangle$ is

f.i) an eigenket of L^2 with eigenvalue $\hbar^2 l(l+1)$; and f.ii) an eigenket of L_z with eigenvalue $\hbar(m-1)$.

Thus, L_- lowers the z-component of angular momentum by $\hbar$.

g) Show, by a suitable choice of phase, that

g.i) $L_-|l,m\rangle = \hbar\sqrt{l(l+1)-m(m-1)}|l,m-1\rangle$; and

g.ii) $L_+|l,m\rangle = \hbar\sqrt{l(l+1)-m(m+1)}|l,m+1\rangle$.

h) Prove that the inequality $-l \leq m \leq l$ will not be violated if

h.i) $2l$ is an integer greater than or equal to zero; and

h.ii) $m = -l, -l+1, \ldots l-1, l$.

j) State the orthonormality condition on $\{|l,m\rangle\}$.

k) Consider a wave function $\psi(\theta,\phi)$ that describes the quantum mechanics of a particle moving on the surface of a unit sphere. A suitable basis set is $\chi_{lm}(\theta,\phi) \equiv \langle\theta,\phi|l,m\rangle$. Show that if $\psi(\theta,\phi)$ is single-valued, then

$$|\psi\rangle = \sum_l \sum_m \psi_{lm}|l,m\rangle$$

can only include terms with integral value of m. What does this imply about allowed values of the angular momentum quantum number l?

l) State the orthonormality condition for $\{|\theta,\phi\rangle\}$.

2) Unit angular momentum:

a) Use the angular momentum raising and lowering operators $L_\pm$ to construct the two 3×3 matrices that represent $L_\pm$ in the $l=1$ sector and in the L_z basis.

b) Use the matrices found in part (a) to construct the matrices that represent L_x and L_y in the same sector. Write down the matrix that represents L_z, and compute the matrix that represents L^2.

c) Show, by calculating commutators, that your three matrices representing L_x, L_y and L_z obey the angular momentum commutation relations.

3) Particle confined to a disk: Solve the energy eigenproblem (*i.e.*, find the energy eigenvalues and eigenfunctions) for a particle confined to a disk of radius d.

4) Orbital angular momentum (after Shankar, 12.3.3): A particle moving in two dimensions is described by the wave function

$$\psi(\rho,\phi) = A\,\mathrm{e}^{-\rho^2/2\Delta^2}\cos^2\phi,$$

where ρ and ϕ are plane polar coordinates and Δ is a real parameter. Show that the probabilities of observing the z component of angular momentum and finding the results $0\hbar$, $2\hbar$ and $-2\hbar$ are, respectively, 2/3, 1/6 and 1/6. Note that it is unnecessary to compute any radial integrals.

5) Spherical harmonics – optional but strongly recommended: In this problem, we will set up and solve the angular momentum eigenproblem in the $|\theta, \phi\rangle$ representation.

First we find representations of the operators $\mathbf{L}$ that act on scalar functions $\psi(\theta, \phi)$. To do this, recall from class that the operators $\mathbf{L}$ act on bras in the following way:

$$\langle \mathbf{r}| \, e^{-\mathbf{a}\cdot\mathbf{L}/i\hbar} = \langle e^{i a_j \ell^j}\, \mathbf{r}|,$$

where summation is implied over $j = 1, \ldots 3$. The rotation matrix $e^{i a_j \ell^j}$ is built by exponentiating the matrix $a_j \ell^j$, which itself is a linear combination of the three matrices ℓ^j with coefficients a_j. The matrices ℓ^j are the generators of rotations of ordinary three-dimensional vectors and have matrix elements $(\ell^j)_{kn} = i\epsilon_{jkn}$.

a) Assume $|\mathbf{a}|$ is infinitesimal, and expand the exponential function to obtain

$$\langle \mathbf{r}| - \frac{1}{i\hbar}\langle \mathbf{r}|(\mathbf{a}\cdot\mathbf{L}) \simeq \langle (\mathbf{r} + \mathbf{a}{\times}\mathbf{r})|.$$

b) Make a Taylor expansion of the right hand side to show that

$$\langle \mathbf{r}|\mathbf{a}\cdot\mathbf{L} = \frac{\hbar}{i}(\mathbf{a}{\times}\mathbf{r})\cdot\frac{\partial}{\partial \mathbf{r}}\,\langle \mathbf{r}|,$$

where $\partial/\partial\mathbf{r}$ is an alternative notation for the gradient operator ∇.

c) Rewrite the operator

$$\frac{\hbar}{i}(\mathbf{a}{\times}\mathbf{r})\cdot\frac{\partial}{\partial \mathbf{r}}$$

in such a way that definition, in terms of angular momentum and angular momentum, becomes apparent.

d) Now return to your answer to part (a). Choose $\mathbf{a}$ to lie along the z-axis, so that the rotation is about the z-axis. By working in spherical polar coordinates, (r, θ, ϕ), find the changes in (r, θ, ϕ) when $\mathbf{r} \to \mathbf{r} + \mathbf{a}{\times}\mathbf{r}$. Hence, show that

$$\langle r, \theta, \phi| L_z = -i\hbar\frac{\partial}{\partial\phi}\langle r, \theta, \phi|.$$

e) Now consider a rotation around the x-axis, again with infinitesimal $|\mathbf{a}|$. Find the changes induced in (r, θ, ϕ) under the rotation $\mathbf{r} \to \mathbf{r} + \mathbf{a}{\times}\mathbf{r}$. Hence, show that

$$\langle r, \theta, \phi| L_x = i\hbar\{\sin\phi\frac{\partial}{\partial\theta} + \cot\theta\cos\phi\frac{\partial}{\partial\phi}\}\langle r, \theta, \phi|.$$

f) Similarly, show that

$$\langle r, \theta, \phi| L_y = i\hbar\{-\cos\phi\frac{\partial}{\partial\theta} + \cot\theta\sin\phi\frac{\partial}{\partial\phi}\}\langle r, \theta, \phi|.$$

g) Hence, show that

$$\langle r,\theta,\phi|L_{\pm} = -i\hbar e^{\pm i\phi}\{\pm i\frac{\partial}{\partial\theta} - \cot\theta\frac{\partial}{\partial\phi}\}\langle r,\theta,\phi|.$$

h) Recall that $L^2 = L_+L_- + L_z^2 - \hbar L_z$. Using this result, show that

$$\langle r,\theta,\phi|L^2 = (i\hbar)^2\{\frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2} + \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\sin\theta\frac{\partial}{\partial\theta}\}\langle r,\theta,\phi|.$$

i) By recalling the definition of ∇^2 in spherical polar coordinates, show that

$$\nabla^2 f(r,\theta,\phi) = \{\frac{1}{r}\frac{\partial^2}{\partial r^2}r - \frac{1}{\hbar^2 r^2}L^2\}f(r,\theta,\phi).$$

j) Show that, together,

$$\begin{aligned} L_+|l,l\rangle &= 0; \\ L_z|l,l\rangle &= \hbar l|l,l\rangle; \end{aligned}$$

imply that $\langle\theta,\phi|l,l\rangle \propto \exp(il\phi)\sin^l\theta$.

k) Briefly describe how you might construct $\langle\theta,\phi|l,m\rangle$ from $\langle\theta,\phi|l,l\rangle$.