Physics 580 | **Quantum Mechanics I** | Prof. P. M. Goldbart
Handout 4 | `http://w3.physics.uiuc.edu/~goldbart` | 2115 & 3135 ESB
26 August 2008 | HOMEWORK 0 | University of Illinois

This particular homework assignment is **optional**, but I encourage you to work through at least some of the problems, especially numbers **2** and **10**. The next homework assignment will be issued on Tuesday September 2.

1) Calculus of variations – straight lines:

a) By using the calculus of variations with fixed end-points, show that the shortest path between two fixed points in a plane is a straight line.

b) By using the calculus of variations with fixed end-points, for the case of several dependent variables, show that the shortest path between two points in three dimensional space is given by a straight line.

c) Reconsider the derivation of the Euler equation for a single dependent variable, now allowing variations that do not necessarily vanish at the ends. By paying particular attention to the boundary terms, show that in the plane the shortest path between a straight line and a point is the perpendicular from the point to the line.

[Note: For the purposes of this question you need only establish that such distances are *stationary* with respect to small variations, and not necessarily (even local, let alone global) minima.]

2) The brachistochrone problem: In this question we will use the calculus of variations to solve a famous old problem, the brachistochrone (or least time) problem.

Let A and B be two points in a vertical plane, with A higher than B. Suppose that A and B are connected by a smooth wire that describes some curve in the vertical plane, and that the downward gravitational acceleration is g. A bead starts from rest at the point A and is allowed to slide to the point B. T, the time taken to slide from A to B, depends on the shape of the curve. By following the steps outlined below, find the curve that makes T stationary.

a) Let y be the depth below A and let x be the horizontal displacement from A. Then A is the point $(x, y) = (0, 0)$, and B is the point $(x, y) = (\overline{x}, \overline{y})$. Construct a functional T, which depends on g and on the curve of the wire $y(x)$, that gives the time taken to reach B. Write T in the form

$$T[y] = \frac{1}{\sqrt{2g}} \int_0^{\overline{x}} dx\, f(y, y'),$$

where y' denotes dy/dx, and write down the appropriate function f.

b) Derive the Euler-Lagrange equation for the curve $y(x)$ that makes T stationary. Of what order is this differential equation?

c) The solution of this problem is simplified by recognising that the function f does not depend on x. To see the simplification, consider a general functional of the form

$$U[y] = \int_0^{\overline{x}} dx\ h(y, y').$$

Write down the Euler-Lagrange equation for this problem, in terms of the function $h(y, y')$. Show that, because $\partial h/\partial x = 0$, we can immediately integrate the Euler-Lagrange equation to obtain the so-called first integral

$$h - y' \frac{\partial h}{\partial y'} = \text{constant}.$$

d) Derive the first integral for the brachistochrone problem. Of what order is this differential equation? What information that we have not yet used will allow us—at a later stage—to fix the constant of integration?

A cycloid is a curve generated by a point on the radius of a wheel of radius C rolling under the x-axis and is specified parametrically by the equations

$$\begin{aligned} x(\phi) &= C(\phi - \sin\phi), \\ y(\phi) &= C(1 - \cos\phi). \end{aligned}$$

e) Show that the cycloid solves the brachistochrone problem. What information would you use to fix the constant C?

f) Discuss briefly, without any calculations, how you might set about establishing that the cycloid not only makes T stationary, but actually minimises it.

3) Foucault's pendulum in disguise: A particle of mass μ moving in three dimensions is bound to the origin $\mathcal{O}$ by a harmonic spring of spring constant κ. Let $\mathbf{R}(t)$ denote the position of the particle at time t.

a) Write down the lagrangian that controls the motion of the system.

Now suppose that the motion is confined to a certain moving plane that passes through $\mathcal{O}$ and has unit normal vector $\mathbf{N}(t)$.

b) Write down the appropriate equation of constraint, and use it to construct the appropriate new lagrangian, which involves a lagrangian undetermined multiplier λ.

c) Construct the classical eqation of motion in terms of λ.

Now restrict your attention to the situation in which the orientation of the plane varies slowly, compared with the natural frequency ω ($\equiv \sqrt{\kappa/\mu}$) of the oscillating particle, *i.e.*, $|\dot{\mathbf{N}}(t)| \ll \omega$. In addition, consider only "linearly polarised" motions, *i.e.*, those that pass through $\mathcal{O}$.

d) By assuming that the only consequence of the motion of the constraint-plane is the slow variation of the direction of oscillation $\mathbf{A}(t)$, ie, that $\mathbf{R}(t) = \mathbf{A}(t)\sin(\omega t + \varphi)$, show that the oscillation-direction $\mathbf{A}(t)$ obeys

$$\dot{\mathbf{A}}(t) \approx \Big(\mathbf{N}(t) \times \dot{\mathbf{N}}(t)\Big) \times \mathbf{A}(t).$$

e) Show that the magnitude of $\mathbf{A}(t)$ does not vary with time.

f) Show that $\mathbf{A}(t)$ is not integrable, *i.e.*, that $\mathbf{A}(t)$ cannot be written as $\mathbf{A}(t) = \mathbf{f}(\mathbf{N}(t))$.

g) Suppose that $\mathbf{N}(t)$ is slowly driven around a closed path over a time T, *i.e.*, $\mathbf{N}(T) = \mathbf{N}(0)$. Find a relationship between $\mathbf{A}(T) \cdot \mathbf{A}(0)/|\mathbf{A}(T)|\,|\mathbf{A}(0)|$ and the area of the unit sphere enclosed by the path $\mathbf{N}(t)$.

h) By using the equation given in part (d) and your answer to parts (g), explain why your answer to part (g) can be described as *geometric*.

[Hint: See the article entitled *The Quantum Phase, Five Years After*, by M. V. Berry, in *Geometric Phases in Physics*, A. Shapere and F. Wilczek (World Scientific, Singapore, 1989), especially p. 8 *et seq*.]

4) Flowing river: A river has parallel straight banks, given by the lines $x = 0$ and $x = b(> 0)$. The velocity $\mathbf{V}$ at which the river water flows is always directed parallel to the banks, but varies with the distance from the banks: $\mathbf{V} = A(x)\,\mathbf{e}_y$, where $A(x)$ is a certain prescribed function. A boat moves at constant speed $C(> |\mathbf{V}|)$ relative to the water, and follows the path $\mathbf{R} = x\,\mathbf{e}_x + y(x)\,\mathbf{e}_y$. Construct the functional $T[y]$ that gives the time to cross the river in terms of the path taken [*i.e.*, $y(x)$ and its derivative(s)], the boat speed C, and the river speed $A(x)$. Constructing an Euler equation is *not* required.

[Hint: Observe that the boat velocity relative to the banks has the form $(dx/dt, dy/dt) = (C\sin\alpha, A + C\cos\alpha)$.]

5) Fermat's principle: According to Fermat's principle, the path taken by a ray of light between any two points makes stationary the travel time between those points. (In this problem you may assume that all light paths lie in a suitable plane.) A medium may be characterised optically by its refractive index n, such that the speed of light in the medium is c/n.

a) Use Fermat's principle to show that light propagates along straight lines in homogeneous media (*i.e.*, media in which n is independent of position).

b) Consider the propagation of light from one semi-infinite homogeneous medium of refractive index n_1 to another with refractive index n_2. By examining paths that need not be differentiable at the interface establish Snell's law.

c) A planar light ray propagates in an inhomogeneous medium with refractive index $n(r)$, where r is the distance to a fixed centre, $\mathcal{O}$. Let r and ϕ be plane polar coordinates.

Use Fermat's principle to find the first-order nonlinear differential equation obeyed by the light path. Choose the constant of integration to be the distance a from $\mathcal{O}$ at which $dr/d\phi = 0$.

d) If $n(r) = \sqrt{1 + \alpha^2/r^2}$ and the ray starts far from $\mathcal{O}$, find the ultimate angle through which the ray is refracted if its minimum distance from $\mathcal{O}$ is a. (The concept of *apsidal angles*, familiar from classical mechanics, may be useful here.) Sketch the path of the ray.
[Hint: you may find helpful the substitution $r = a/\sin\psi$.]

6) Mass distribution for prescribed profile: You are provided with a light-weight line of length $\pi a/2$ and some lead shot of total mass M. Determine how the lead should be distributed along the line if the loaded line is to hang in an arc of a circle of radius a when its ends are attached to two points at the same height.

7) Mechanical equilibrium of a hard ferromagnet: Let $\mathbf{m}(\mathbf{x})$ be the local value of the magnetisation in a ferromagnet. Suppose that the ferromagnet is *hard*, which means that the *magnitude* of the magnetisation is everywhere equal to unity. Suppose, further, that the free energy of the ferromagnet is given by

$$E = \frac{1}{2} J \int_V dV \left(\partial_\mu m^a(\mathbf{x})\right)^2 \equiv \frac{1}{2} J \int_V dV \sum_{\mu=1}^{3} \sum_{a=1}^{3} \left(\partial_\mu m^a(\mathbf{x})\right)^2 ,$$

where V denotes the volume of the sample.

a) By making a small variation of the magnetisation which vanishes at the boundary of the sample and integrating by parts, and by using a Lagrange multiplier at each position $\mathbf{x}$ to enforce the (non-linear) constraint that $|\mathbf{m}(\mathbf{x})| = 1$, show that the condition for equilibrium is given by

$$\nabla^2 m^a(\mathbf{x}) - m^a(\mathbf{x})\, m^b(\mathbf{x}) \nabla^2 m^b(\mathbf{x}) = 0 \,.$$

b) What do you think is the origin of the non-linearity in this partial differential equation?

c) Briefly discuss the number of independent partial differential equations, in the context of the number of dependent variables.

8) Ginzburg-Landau theory of superconductivity from a variational principle: Consider the increase F in the thermodynamic free energy due to a nonzero complex Ginzburg-Landau order parameter $\Psi(\mathbf{r})$ and magnetic vector potential $\mathbf{A}(\mathbf{r})$ which, in general, depend on the position in the sample $\mathbf{r}$. In the Ginzburg-Landau theory, this is given by

$$F = \int d^3r \left\{ \frac{1}{8\pi} |\mathbf{B}|^2 + \frac{\hbar^2}{4m} | \left(\nabla - \frac{2ie}{\hbar c} \mathbf{A} \right) \Psi|^2 + a|\Psi|^2 + \frac{1}{2} b |\Psi|^4 \right\} .$$

Here, the magnetic induction $\mathbf{B}$ is given by $\nabla \times \mathbf{A}$, m is the electron effective mass, e is the electon charge, $\hbar$ is Planck's constant, c is the speed of light *in vacuo*, and a and b are parameters that characterize the thermodynamics of superconductivity.

By demanding that F be stationary with respect to independent variations of the complex order parameter and the real vector potential, including ones that can be nonzero at the sample boundary, show that the classical values of the order parameter and vector potential obey the coupled Ginzburg-Landau/Ampère system:

$$\begin{aligned} \frac{1}{4m}\left(-i\hbar\nabla - \frac{2e}{c}\right)^2 \Psi + a\Psi + b|\Psi|^2\Psi &= 0, \\ \nabla \times \mathbf{B} &= \frac{4\pi}{c}\mathbf{j}, \end{aligned}$$

where $\mathbf{j}$ is given by

$$\mathbf{j} = -\frac{ie\hbar}{2m}\left(\Psi^*\nabla\Psi - \Psi\nabla\Psi^*\right) - \frac{2e^2}{mc}|\Psi|^2\mathbf{A},$$

together with the natural boundary conditions

$$\mathbf{n}\cdot\left(-i\hbar\nabla - \frac{2e}{c}\mathbf{A}\Psi\right) = 0$$

are obeyed at the superconductor interface between the sample and the vacuum that surrounds it, the outward unit normal to which is $\mathbf{n}$.

By considering the variational derivative $\delta F/\delta\mathbf{A}(\mathbf{r})$, show that $\mathbf{j}$ is the charge current density.

9) Form-invariance of the Euler-Lagrange equation: In this question we are going to consider the Euler-Lagrange equation (ELE) for a single dynamical degree of freedom q. (The argument can readily be generalised to the case of many degrees of freedom.)

Suppose that a variable q satisfies the ELE that follows from a lagrangean $\mathcal{L}(q,\dot{q},t)$. Suppose, now, that we make a change of variable from q to $r(q)$. Show that, by virtue of the original ELE for q, the new variable r satisfies the ELE that follows from the new lagrangean $\mathcal{M}(r,\dot{r},t)$, where $\mathcal{M}(r,\dot{r},t)$ is given by

$$\mathcal{M}(r,\dot{r},t) = \mathcal{L}(q, \frac{\partial q}{\partial r}\dot{r}, t).$$

This result shows that the *form* of the ELE is invariant (or preserved) under the transformation $q \to r(q)$. As we shall see, the hamiltonian formalism admits even more general transformations, *i.e.*, those that mix coordinates and momenta and yet leave the form of Hamilton's equations invariant.

10) Cartesian vectors: The aim of this question is to familiarise you with some of the notational conventions that we shall be using throughout the course, and also to give you some practice with the so-called summation convention, due to Einstein.

Consider a *cartesian basis*, $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$, for 3-dimensional vectors $\mathbf{x}$. Suppose that the basis vectors are normalised to unity and are mutually orthogonal (*i.e.*, they are orthonormal); then they possess the scalar products $\mathbf{e}_\mu \cdot \mathbf{e}_\nu = \delta_{\mu\nu}$, where μ and ν take the values 1, 2, or 3 (or x, y or z). Here, $\delta_{\mu\nu}$ is the Kronecker symbol, which equals 1 when $\mu = \nu$, and equals 0 otherwise. You may think of this as the (3×3) identity matrix

$$\begin{pmatrix} \delta_{11} & \delta_{12} & \delta_{13} \\ \delta_{21} & \delta_{22} & \delta_{23} \\ \delta_{31} & \delta_{32} & \delta_{33} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

An arbitrary vector $\mathbf{x}$ is a linear combination of basis vectors, $\mathbf{x} = \sum_{\mu=1}^{3} x_\mu \mathbf{e}_\mu$, with the set of coefficients (called components) $\{x_\mu\}_{\mu=1}^{3}$. Notice that we can extract a component by taking the scalar product of a vector with the appropriate basis vector,

$$\mathbf{e}_\mu \cdot \mathbf{x} = \mathbf{e}_\mu \cdot \sum_{\nu=1}^{3} x_\nu \mathbf{e}_\nu = \sum_{\nu=1}^{3} \mathbf{e}_\mu \cdot \mathbf{e}_\nu x_\nu = \sum_{\nu=1}^{3} \delta_{\mu\nu} x_\nu = x_\mu.$$

It is very useful to adopt a convention, called summation convention, in which summation is implied over any twice-repeated indices; *e.g.*,

$$\mathbf{x} = \sum_{\mu=1}^{3} x_\mu \mathbf{e}_\mu \equiv x_\mu \mathbf{e}_\mu.$$

In true tensorial equations a given index, say μ, never need occur more than twice. Singly occurring indices are called effective indices, whilst repeated indices are called dummy indices, and may be replaced by another index: *e.g.*, $\mathbf{x} = x_\nu \mathbf{e}_\nu = x_\mu \mathbf{e}_\mu$. Dummy indices are rather like dummy variables in integrals.

If two vectors $\mathbf{a}$ and $\mathbf{b}$ are equal then their components are equal, *i.e.*, $a_\mu = b_\mu$. This follows from taking the scalar product of both sides of the equation $\mathbf{a} = \mathbf{b}$ with the basis vector $\mathbf{e}_\mu$. Notice that unrepeated indices balance throughout all terms of an equation. For example, if $\mathbf{a} + \mathbf{b} = \mathbf{c}$ then $a_\mu + b_\mu = c_\mu$. Indices are only considered repeated if they occur in the *same term*. For example, the equation $a_\mu = b_\mu$ contains one effective index and no repeated indices. Using $\mathbf{e}_\mu \cdot \mathbf{e}_\nu = \delta_{\mu\nu}$ and $\mathbf{e}_\mu \cdot \mathbf{x} = x_\mu$, and also the definition of $\delta_{\mu\nu}$, verify the following statements:

a.1) $\mathbf{x} \cdot \mathbf{x} = x_\mu x_\mu$

a.2) $\mathbf{x} \cdot \mathbf{y} = x_\mu y_\mu$

a.3) $\delta_{\mu\nu}\, \delta_{\nu\rho} = \delta_{\mu\rho}$

a.4) $a_\mu = a_\nu \delta_{\nu\mu}$

a.5) $\delta_{\mu\mu} = 3$.

Now consider scalar and vector fields, *i.e.*, scalar-valued functions, $f(\mathbf{x})$, and vector-valued functions, $\mathbf{g}(\mathbf{x}) = \mathbf{e}_\mu \, g_\mu(\mathbf{x})$, of a position vector, $\mathbf{x}$. For cartesian coordinates, the gradient operator ∇ is defined by

$$\nabla \equiv \sum_{\mu=1}^{3} \mathbf{e}_\mu \, \frac{\partial}{\partial x_\mu} = \mathbf{e}_\mu \, \frac{\partial}{\partial x_\mu} = \mathbf{e}_\mu \, \partial_\mu$$

where, for convenience, we have written ∂_μ for $\partial/\partial x_\mu$.
Verify the following results:

b.1) $\nabla \cdot \mathbf{g}(\mathbf{x}) = \partial_\mu g_\mu(\mathbf{x})$
b.2) $\nabla f(\mathbf{x}) = \mathbf{e}_\mu \partial_\mu f(\mathbf{x})$
b.3) $(\mathbf{x} \cdot \nabla) \, f(\mathbf{x}) = x_\mu \, \partial_\mu f(\mathbf{x})$
b.4) $\nabla \cdot (\nabla f(\mathbf{x})) = \nabla^2 f(\mathbf{x})$ where $\nabla^2 \equiv \partial_x^2 + \partial_y^2 + \partial_z^2 = \partial_\mu \partial_\mu$
b.5) $\nabla \cdot \mathbf{x} = \partial_\mu \, x_\mu = \delta_{\mu\mu} = 3$
b.6) $\nabla(\mathbf{x} \cdot \mathbf{x}) = 2 \, \mathbf{x}$
b.7) $\nabla^2 \, (\mathbf{x} \cdot \mathbf{x}) = 6$
b.8) $\nabla |\mathbf{x}| = \mathbf{x}/|\mathbf{x}|$
b.9) $\partial_\mu (x_\nu/|\mathbf{x}|) = (x^2 \delta_{\mu\nu} - x_\mu \, x_\nu)/x^3$.
b.10) $\nabla^2 \, (1/|\mathbf{x}|) = -4\pi \, \delta(\mathbf{x})$ [Hint: Apply the divergence theorem.]
b.11) $\nabla \, (\mathbf{x} \cdot \mathbf{g}(\mathbf{x})) = \mathbf{g}(\mathbf{x}) + \mathbf{e}_\mu x_\nu \partial_\mu g_\nu(\mathbf{x})$
b.12) $\nabla \cdot (\mathbf{x} \, f(\mathbf{x})) = 3 \, f(\mathbf{x}) + (\mathbf{x} \cdot \nabla) \, f(\mathbf{x})$
b.13) For constant $\mathbf{h}$, $\oint_\Gamma d\mathbf{x} \cdot (\frac{1}{2}\mathbf{h} \times \mathbf{x}) = \pi \mathbf{h} \cdot \mathbf{n}$, where Γ is a any circle of unit radius, and the unit vector $\mathbf{n}$ specifies the axis of the circle and the sense in which it is traversed. [Hint: Apply the Stokes theorem.]
b.14) $\nabla \, \exp(i\mathbf{k} \cdot \mathbf{x}) = i\mathbf{k} \, \exp(i\mathbf{k} \cdot \mathbf{x})$
b.15) $f(\mathbf{x} + \mathbf{a}) = f(\mathbf{x}) + (\mathbf{a} \cdot \nabla) f(\mathbf{x}) + \cdots = \mathrm{e}^{\mathbf{a} \cdot \nabla} f(\mathbf{x})$. [Note: This is a compact form of the multidimensional Taylor theorem.]

A large selection of further practice problems can be found in: M. R. Spiegel, *Vector Analysis* (McGraw-Hill, 1974), especially Chapters 4–6.