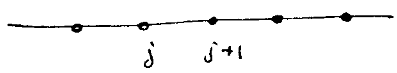
Ref.: Schultz, Mattis and Lieb, Rev. Mod. Physics $\underline{36}$, 856 (1964)

* Spin Chains and one-dimensional fermi systems.

Consider a one-dimensional chain with N sites an a spin one-half degree of freedom sitting at each site. As an example we will ~~the~~ discuss the case of the Ising Model in a transverse field. This model is closely connected with the two-dimensional Ising model of Statistical Mechanics.

The Hamiltonian is

$$H = -\sum_{j=1}^{N} \sigma_1(j) - \lambda \sum_{j=1}^{N} \sigma_3(j)\,\sigma_3(j+1)$$



Throughout we'll assume periodic boundary conditions, i.e. $\sigma_3(N+1) \equiv \sigma_3(1)$

This model has strikingly different behavior for $\lambda >> 1$ and $\lambda << 1$

For $\lambda >> 1$ the ground state is $|\underset{1}{\uparrow} \cdots\cdots \underset{N}{\uparrow}\rangle$ in the representation where $\sigma_3 |s\rangle = s|s\rangle$. There is another ground state available $|\downarrow \cdots\cdots \downarrow\rangle$. Both have the same energy.

$E_{gnd}(\lambda \to \infty) = -\lambda N$ Thus the ground state is doubly degenerate. However as $N \to \infty$ these ground states do not communicate and don't mix to any finite order in perturbation theory. In $\underline{\text{any}}$ of these ground states

$$\langle gnd|\, \sigma_3(l)\, |gnd\rangle \underset{\lambda \to \infty}{=} \pm 1 \qquad (+\uparrow, -\downarrow)$$

$$\langle gnd|\, \sigma_3(0)\, \sigma_3(l)\, |gnd\rangle_{\lambda \to \infty} = 1$$

In the other limit $(\lambda \to 0)$ the natural representation is that in which $\underline{\sigma_1}$ is diagonal $\Rightarrow$ $\langle gnd|\, \sigma_3(l)\, |gnd\rangle_{\lambda \to 0} = 0$

Thus the system is magnetized for $\lambda >> 1$ and demagnetized for $\lambda << 1$

This system has a symmetry : spin-flip.

Let Q be the generator of ~~this~~ this discrete symmetry

$$Q = \prod_{j=1}^{N} \sigma_1(j) = Q^{-1}$$

$$\Rightarrow \qquad Q^{-1}\, \sigma_3(\ell)\, Q = -\sigma_3(\ell)$$

$\Rightarrow$ ~~Of these~~ and $\quad [H, Q] = 0$

For a finite system this statement implies that $\langle \text{gnd} | A | \text{gnd} \rangle_\lambda = 0$

if ~~Q~~ $[A, Q] \neq 0$. This is not true for the infinite system.

because the ground state $|\text{gnd}\rangle$ may be a state of spontaneously

broken symmetry (as we saw at $\lambda \to \infty$)

$$Q\, |\text{gnd}\rangle \neq |\text{gnd}\rangle$$

because we have a degeneracy ~~$|\text{gnd}\rangle_\uparrow$~~, ~~$|\text{gno}$~~ $|\text{gnd}\rangle_\uparrow$, $|\text{gnd}\rangle_\downarrow$

and $\quad Q\,|\text{gnd}\rangle_\uparrow = |\text{gnd}\rangle_\downarrow \qquad Q\,|\text{gnd}\rangle_\downarrow = |\text{gnd}\rangle_\uparrow$

Physically this is realized by turning on a longitudinal field ~~σ_3~~ $\sum_j h\sigma_3(\ell)$

which breaks the symmetry, take the ∞ Volume limit <u>first</u> and

at the end let $h \to 0$.

The Pauli Spin operators are operators with mixed symmetry.

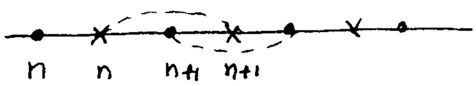
$$[\sigma_3(j),\ \sigma_1(\ell)] = 0 \qquad \ell \neq j \qquad \text{but} \quad \sum \{\sigma_3(j); \sigma_1(j)\} = 0$$

Thus they are "bosons with hard cores". $\qquad \sigma^+(\ell)\,|\downarrow\rangle = |\uparrow\rangle$

$$\sigma^+(\ell)\,|\uparrow\rangle = 0$$

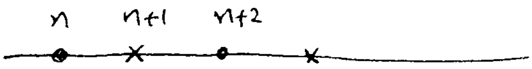
We could proceed to work directly with the model as it stands. However it will be convenient, for further connections, to perform a few minor modifications.

$\boxed{L.16}$ Consider two decoupled independent Ising models in a transverse field, the dots and the crosses. Let's denote the dots by σ's and the crosses by τ's, satisfying a Pauli algebra



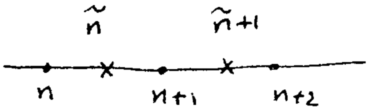
$$\Rightarrow \quad H = -\sum_n \left[\sigma_1(n) + \tau_1(n)\right] - \lambda \sum_n \left[\sigma_3(n)\sigma_3(n+1) + \tau_3(n)\tau_3(n+1)\right]$$

We may regard these as part of one single lattice. The dots on even sites, the crosses on odd sites. And we'll denote the operators just by σ's.



$$\Rightarrow \quad H = -\sum_n \left[\sigma_1(2n) + \sigma_1(2n+1)\right] - \lambda \sum_n \left[\sigma_3(2n)\,\sigma_3(2n+2) + \sigma_3(2n+1)\sigma_3(2n+1)\right]$$

Dual transformation: Define a new set of Pauli operators sitting on the dual lattice sites.



$$\begin{cases} \tau_1(\tilde{n}) = \sigma_3(n)\,\sigma_3(n+1) \\ \tau_3(\tilde{n}) = \prod_{m \leq \tilde{n}} \sigma_1(m) \\ i\,\tau_2(\tilde{n}) = \tau_3(\tilde{n})\,\tau_1(\tilde{n}) \end{cases}$$

It is easy to prove that τ_1, τ_3 anticommute $\{\tau_1(\tilde{n}), \tau_3(\tilde{n})\} = 0$

$$[\tau_1(\tilde{n}), \tau_3(\tilde{m})] = 0$$

Note $\tau_3(\tilde{n})$ is a kink-creation operator $\uparrow\uparrow\uparrow\uparrow_{\underset{\tilde{n}}{\times}}\uparrow\uparrow\uparrow\uparrow \rightarrow \downarrow\downarrow\downarrow\downarrow_{\underset{\tilde{n}}{\times}}\uparrow\uparrow\uparrow\uparrow\uparrow$

$$\Rightarrow \quad \tau_3(\tilde{n})\,\tau_3(\tilde{n}+1) = \sigma_1(n+1)$$

$$\tau_3(\tilde{n}-1)\,\tau_3(\tilde{n}) = \sigma_1(n)\,\sigma_1(n+1)$$

$$\sigma_3(n) = \prod_{\tilde{m}<\tilde{n}} \tau_1(\tilde{m}) \Rightarrow \sigma_3(n)\,\sigma_3(n+2) = \tau_1(\tilde{n})\,\tau_1(\tilde{n}+1)$$

$$H = -\sum_{\tilde n}\left[\tau_3(\tilde n)\tau_3(\tilde n+1) + \lambda\, \tau_1(\tilde n)\tau_1(\tilde n+1)\right]$$

or

$$H = -\sum_{\tilde n}\left(\frac{1+\lambda}{2}\right)\left[\tau_3(\tilde n)\tau_3(\tilde n+1) + \tau_1(\tilde n)\tau_1(\tilde n+1)\right] -$$

$$-\sum_{\tilde n}\left(\frac{1-\lambda}{2}\right)\left[\tau_3(\tilde n)\tau_3(\tilde n+1) - \tau_1(\tilde n)\tau_1(\tilde n+1)\right]$$

by changing the scale of energies $H \to \left(\frac{1+\lambda}{2}\right) H$

$$\Rightarrow H = -\sum_{\tilde n}\left(\tau_3(\tilde n)\tau_3(\tilde n+1) + \tau_1(\tilde n)\tau_1(\tilde n+1)\right) - \left(\frac{1-\lambda}{1+\lambda}\right)\sum_{\tilde n}\left[\tau_3(\tilde n)\tau_3(\tilde n+1) - \tau_1(\tilde n)\tau_1(\tilde n+1)\right]$$

Jordan - Wigner transformation

We want to transform a Pauli algebra into a Fermi algebra. Thus we want a set of operators that anticommute at different points.

Let $K(\tilde n)$ be the kink creation operator

$$K(\tilde n) = \prod_{\tilde m < \tilde n}\left(-\tau_2(\tilde m)\right)$$

(the - sign is there for later convenience

$$\Rightarrow \chi(\tilde n) = \left[\prod_{\tilde m<\tilde n}\left(-\tau_2(\tilde m)\right)\right]\tau_3(\tilde n)$$

$$\Rightarrow \{\chi(\tilde n),\ \chi(\tilde n')\} = 0$$ since $\{\tau_3(\tilde n),\ \tau_2(\tilde n)\}$

Same thing with

$$\chi'(\tilde n) = \left[\prod_{\tilde m<\tilde n}\left(-\tau_2(\tilde m)\right)\right]\tau_1(\tilde n)$$

$$\{\chi'(\tilde n),\ \chi'(\tilde n')\} = \{\chi'(\tilde n),\ \chi(\tilde n')\} = 0$$

(continued)

Hence, they are almost fermions, although not quite since

$$\chi(\tilde{n})^2 = \chi'(\tilde{n})^2 = 1$$

If we define

$$\tau^{\pm}(\tilde{n}) = \frac{1}{2}\left(\tau_3(\tilde{n}) \pm i\,\tau_1(\tilde{n})\right)$$

$$\Rightarrow \begin{cases} \psi^{+}(\tilde{n}) = \frac{1}{2}\left(\chi(\tilde{n}) + i\,\chi'(\tilde{n})\right) \\ \psi(\tilde{n}) = \frac{1}{2}\left(\chi(\tilde{n}) - i\,\chi'(\tilde{n})\right) \end{cases}$$

<u>are fermion operators</u>

will have the required property

$$\{\psi(\tilde{n}), \psi(\tilde{m})\} = 0 \qquad \tilde{n} \neq \tilde{m}$$

$$\psi^{+}(\tilde{n})^2 = 0 = \frac{1}{4}\left[1 - 1 + i\,\{\chi(\tilde{n}), \chi'(\tilde{n})\}\right]$$

$$\psi^{+}(\tilde{n}) = K(\tilde{n})\,\tau^{+}(\tilde{n})$$

$$\psi(\tilde{n}) = K^{+}(\tilde{n})\,\tau^{-}(\tilde{n})$$

$$K^{+} = K$$

and

$$\{\psi(\tilde{n}), \psi^{+}(\tilde{n})\} = \{\tau^{-}(\tilde{n}), \tau^{+}(\tilde{n})\} = 1$$

also:

$$\tau^{+}(\tilde{n})\,\tau^{-}(\tilde{n}) = \frac{1}{2}\left(1 + \tau_2(\tilde{n})\right) \Rightarrow$$

$$\{\tau^{+}, \tau^{-}\} = 1$$

$$[\tau^{+}, \tau^{-}] = \tau_2$$

$$\Rightarrow \; -\tau_2(\tilde{n}) = 1 - 2\,\tau^{+}(\tilde{n})\,\tau^{-}(\tilde{n})$$

and $\psi^{+}(\tilde{n})\,\psi(\tilde{n}) = \tau^{+}(\tilde{n})\,\tau^{-}(\tilde{n})$

$$\Rightarrow \; -\tau_2(\tilde{n}) = 1 - 2\,\psi^{+}(\tilde{n})\,\psi(\tilde{n}) = e^{i\pi\,\psi^{+}(\tilde{n})\,\psi(\tilde{n})}$$

since $\psi^{+}\psi = 0$

$$\Rightarrow \; -\tau_2(\tilde{n}) = e^{i\pi\,\psi^{+}(\tilde{n})\,\psi(\tilde{n})}$$

$$\Rightarrow \boxed{K(\tilde{n}) = \prod_{\tilde{m}<\tilde{n}}\left[-\tau_2(\tilde{m})\right] = e^{i\pi\sum_{\tilde{m}<\tilde{n}}\psi^{+}(\tilde{m})\,\psi(\tilde{m})}} = K^{+}(\tilde{n})$$

and

$$\tau_3(\tilde{n}) = \tau^{+}(\tilde{n}) + \tau^{-}(\tilde{n}) = \left(\psi^{+}(\tilde{n}) + \psi(\tilde{n})\right) \exp\left[i\pi\sum_{\tilde{m}<\tilde{n}}\psi^{+}(\tilde{m})\psi\right.$$

$$\tau_1(\tilde{n}) = \frac{1}{i}\left(\tau^{+}(\tilde{n}) - \tau^{-}(\tilde{n})\right) = \frac{1}{i}\left(\psi^{+}(\tilde{n}) - \psi(\tilde{n})\right) \exp\left[i\pi\sum_{\tilde{m}<\tilde{n}}\psi^{+}\psi\right.$$

$$\tau_1(\tilde{n}) = \frac{1}{i}\left[\psi^+(\tilde{n}) - \psi(\tilde{n})\right] \exp\left[i\pi \sum_{\tilde{m}<\tilde{n}} \psi^+(\tilde{m})\psi(\tilde{m})\right]$$

$$\tau_2(\tilde{n}) = 2\,\psi^+(\tilde{n})\,\psi(\tilde{n}) - 1$$

$$\tau_3(\tilde{n}) = \left[\psi^+(\tilde{n}) + \psi(\tilde{n})\right] \exp\left[i\pi \sum_{\tilde{m}<n} \psi^+(\tilde{m})\psi(\tilde{m})\right]$$

We are now ready to rewrite H in terms of fermi operators.

$$\tau_3(\tilde{n})\tau_3(\tilde{n}+1) + \tau_1(\tilde{n})\tau_1(\tilde{n}+1) = 2\left(\tau^+(\tilde{n})\tau^-(\tilde{n}+1) + \tau^-(\tilde{n})\tau^+(\tilde{n}+1)\right) =$$

$$= 2\Big[e^{i\pi\sum_{\tilde{m}<\tilde{n}} \psi^+_{\tilde{m}}\psi_{\tilde{m}}}\,\psi^+_{\tilde{n}}\, e^{i\pi\sum_{\tilde{m}\le\tilde{n}} \psi^+_{\tilde{m}}\psi_{\tilde{m}}}\,\psi_{\tilde{n}+1} +$$

$$+ e^{i\pi\sum_{\tilde{m}<\tilde{n}} \psi^+_{\tilde{m}}\psi_{\tilde{m}}}\,\psi_{\tilde{n}}\, e^{i\pi\sum_{\tilde{m}\le\tilde{n}} \psi^+_{\tilde{m}}\psi_{\tilde{m}}}\,\psi^+_{\tilde{n}+1}\Big] =$$

$$= 2\left[\psi^+_{\tilde{n}}\, e^{i\pi\psi^+_{\tilde{n}}\psi_{\tilde{n}}}\,\psi_{\tilde{n}+1} + \psi_{\tilde{n}}\, e^{i\pi\psi^+_{\tilde{n}}\psi_{\tilde{n}}}\,\psi^+_{n+1}\right] =$$

$$= 2\left[\psi^+_{\tilde{n}}(1 - 2\psi^+_{\tilde{n}}\psi_{\tilde{n}})\psi_{\tilde{n}+1} + \psi_{\tilde{n}}(1 - 2\psi^+_{\tilde{n}}\psi_{\tilde{n}})\psi_{\tilde{n}+1}\right] =$$

$$= 2\left[\psi^+_{\tilde{n}}\psi_{n+1} + \psi_{\tilde{n}}\psi^+_{\tilde{n}+1} - 2\psi_{\tilde{n}}\psi^+_{\tilde{n}+1}\right] =$$

$$= 2\left[\psi^+_{\tilde{n}}\psi_{\tilde{n}+1} - \psi_{\tilde{n}}\psi^+_{\tilde{n}+1}\right] =$$

$$\Rightarrow \tau_3(\tilde{n})\tau_3(\tilde{n}+1) + \tau_1(\tilde{n})\tau_1(\tilde{n}+1) = 2\left[\psi^+(\tilde{n})\psi(\tilde{n}+1) + \psi^+(\tilde{n}+1)\psi(\tilde{n})\right]$$

also

$$\tau_3(\tilde{n})\tau_3(\tilde{n}+1) - \tau_1(\tilde{n})\tau_1(\tilde{n}+1) = 2\left(\tau^+(\tilde{n})\tau^+(\tilde{n}+1) + \tau^-(\tilde{n})\tau^-(\tilde{n}+1)\right) =$$

$$= 2\left(\psi^+(\tilde{n})\psi^+(\tilde{n}+1) + \psi(\tilde{n}+1)\psi(\tilde{n})\right)$$

Thus we get, after rescaling the energies by a factor of 2,

$$H = -\sum_{\tilde{n}}\left(\psi^+(\tilde{n})\psi(\tilde{n}+1) + \psi(\tilde{n}+1)\psi(\tilde{n})\right) - \left(\frac{1-\lambda}{1+\lambda}\right)\sum_{\tilde{n}}\left(\psi^+(\tilde{n})\psi^+(\tilde{n}+1) + \psi(\tilde{n}+1)\psi(\tilde{n})\right)$$

up to a boundary term.

In the last lecture we showed that the spin chain

$$H = -\sum_n \left(\tau_3(n)\tau_3(n+1) + \tau_1(n)\tau_1(n+1)\right) + \left(\frac{\lambda-1}{\lambda+1}\right)\sum_n \left(\tau_3(n)\tau_3(n+1) - \tau_1(n)\tau_1(n+1)\right)$$

is equivalent to a one dimensional fermi system

$$H = -\sum_n \left(\psi^\dagger(n)\,\psi(n+1) + \psi(n+1)\,\psi(n)\right) + \left(\frac{\lambda-1}{\lambda+1}\right)\sum_n \left(\psi^\dagger(n)\,\psi^\dagger(n+1) + \psi(n+1)\,\psi(n)\right)$$

This Hamiltonian does not conserve the total number of particles

and hence $\quad [H, N] \neq 0 \quad$ $\boxed{\text{if} \quad \lambda \neq 1}$

where $\quad N = \sum_n \psi^\dagger(n)\,\psi(n)$

However the operator $Q_\pi = e^{i\pi N}$ does commute with H.

Q_π generates a discrete symmetry (spin-flip)

$$\left.\begin{aligned} Q_\pi^{-1}\,\tau_3\,Q_\pi &= -\tau_3 \\ Q_\pi^{-1}\,\tau_1\,Q_\pi &= -\tau_3 \end{aligned}\right\}$$

At $\lambda = 1 \quad [H, N] = 0$ and N becomes the generator of a continuous symmetry. Let Q_α be $Q_\alpha = e^{i\alpha N} = e^{i\frac{\alpha}{2}\sum_n \tau_2(n)}$

$$\Rightarrow \quad Q_\alpha^{-1}\,\tau_3\,Q_\alpha = \cos\alpha\,\tau_3 + \sin\alpha\,\tau_1 \qquad \text{rotation}$$

$$Q_\alpha^{-1}\,\tau_1\,Q_\alpha = -\sin\alpha\,\tau_3 + \cos\alpha\,\tau_1$$

$$Q_\alpha^{\dagger} H Q_\alpha = H$$

. The symmetry group is $U(1)$.

$$N = \sum_n \psi^\dagger(n)\,\psi(n) = \frac{1}{2}\Big[\sum_n \tau_2(n) + \underset{\substack{\uparrow \\ \#\text{ of sites}}}{M}\Big]$$

$\Rightarrow N \sim \frac{1}{2}\sum_n \tau_2(n) \Rightarrow$ generates rotation with angle α

* Diagonalization of the quadratic form.

$$H = - \sum_n \left(\psi^\dagger(n)\, \psi(n+1) + \psi^\dagger_{(n+1)}\, \psi(n) \right) - \left(\frac{\lambda-1}{\lambda+1}\right) \sum_n \left[\psi^\dagger(n)\, \psi^\dagger(n+1) + \psi(n)\, \psi(n+1) \right]$$

with P.B.C.

(1) Go to momentum space.

$$a(k) = \frac{1}{\sqrt{2M+1}} \sum_{n=-M}^{M} e^{ikn}\, \psi(n)$$

$$k = 0, \pm \frac{2\pi s}{2M+1} \qquad s = 0, \dots, M$$

$$\Rightarrow \quad \{a(k), a(q)^\dagger\} = \delta_{kq}$$

using $\displaystyle \frac{1}{2M+1} \sum_k e^{-ik(n-m)} = \delta_{n,m}$

$$\frac{1}{2M+1} \sum_n e^{in(k-\ell)} = \delta_{k,\ell}$$

$$\psi(n) = \frac{1}{\sqrt{2M+1}} \sum_k e^{-ikn}\, a(k)$$

$$\sum_n \psi^\dagger(n)\, \psi(n+1) = \frac{1}{2M+1} \sum_{k,q} \left[\sum_n e^{i(k-q)n} \right] a^\dagger(k)\, a(q)\, e^{-iq} = \sum_k e^{-ik} a^\dagger(k) a$$

$$\sum_n \psi^\dagger(n+1)\, \psi(n) = \frac{1}{2M+1} \sum_{k,q} \left[\sum_n e^{-i(k-q)n} \right] a^\dagger(q)\, a(k)\, e^{ik} = \sum_k e^{ik} a^\dagger(k)\, a$$

$$\sum_n \psi^\dagger(n)\, \psi^\dagger(n+1) = \frac{1}{2M+1} \sum_{k,q} \left(\sum_n e^{i(k+q)n} \right) a^\dagger(k)\, a^\dagger(q)\, e^{iq} = \sum_k a^\dagger(k)\, a^\dagger(-k)$$

$$\sum_n \psi(n+1)\, \psi(n) = \sum_k a(-k)\, a(k)\, e^{ik}$$

$$H = -\sum_k \left[2\cos k \; a^\dagger(k)\, a(k) + \frac{\lambda-1}{\lambda+1} \left(a^\dagger(k)\, a^\dagger(-k)\, e^{-ik} + a(-k)\, a(k)\, e^{ik} \right) \right]$$

$$H = -2\, a^\dagger(0)\, a(0) - \sum_{k>0} \Big\{ 2\cos k \left[a^\dagger(k)\, a(k) + a^\dagger(-k)\, a(-k) \right] +$$

$$+ \left(\frac{\lambda-1}{\lambda+1} \right) (-2i) \sin k \left[a^\dagger(k)\, a^\dagger(-k) + a(-k)\, a(k) \right] \Big\}$$

$\boxed{\text{L.17}}$ <u>Bogoliubov transformation:</u>

$$\eta(k) = u_k\, a(k) + i\, v_k\, a(-k)^\dagger \quad \Big| \quad \eta(-k) = u_k\, a(-k) - i v_k\, a^\dagger(k) \qquad k > 0$$

$$\eta^\dagger(k) = u_k\, a^\dagger(k) - i\, v_k\, a(-k) \quad \Big| \quad \eta^\dagger(-k) = u_k\, a(-k)^\dagger + i v_k\, a(k)$$

$$\{ \eta(k), \eta^\dagger(k') \} = u_k u_{k'} \{ a(k), a^\dagger(k') \} + v_k v_{k'} \{ a^\dagger(-k), a(-k') \}$$
$$= (u_k^2 + v_k^2)\, \delta_{kk'}$$

$\Rightarrow \quad u_k^2 + v_k^2 = 1 \qquad$ this is enough to ensure that the commutation relations are preserved.

$$\Rightarrow \quad a(k) = u_k\, \eta(k) - i v_k\, \eta^\dagger(-k) \quad \Big| \quad a(-k) = u_k\, \eta(-k) + i v_k\, \eta(k)^\dagger$$

$$a(k)^\dagger = u_k\, \eta(k)^\dagger + i\, v_k\, \eta(-k) \quad \Big| \quad a(-k)^\dagger = u_k\, \eta^\dagger(-k) - i v_k\, \eta(k)$$

By substitution:

$$H = -2\, a^\dagger(0)\, a(0) - \sum_{k>0} \left(\eta^\dagger(k)\, \eta(k) + \eta^\dagger(-k)\, \eta(-k) \right) \left[2\cos k\, (u_k^2 - v_k^2) + \left(\frac{\lambda-1}{\lambda+1} \right) 4 \sin k\, u_k v \right.$$

$$- \sum_{k>0} \left(\eta^\dagger(k)\, \eta^\dagger(-k) + \eta(k)\, \eta(-k) \right) \left[2\cos k\, (-2i)\, u_k v_k + \left(\frac{\lambda-1}{\lambda+1} \right) (-2i) \sin k (u_k^2 \right.$$

$$+ - \sum_{k>0} \left(4\cos k\, v_k^2 + \frac{\lambda-1}{\lambda+1}\, 4 \sin k\, u_k v_k \right)$$

* Lecture 29 (11/4)

We now demand that the fermion non-conserving pieces vanish.

$$2 u_k v_k \cos k \mp \left(\frac{\lambda-1}{\lambda+1}\right) \sin k \,(u_k^2 - v_k^2) = 0$$

$$u_k^2 + v_k^2 = 1$$

$$\Rightarrow \left.\begin{aligned} u_k &= \cos\theta_k \\ v_k &= \sin\theta_k \end{aligned}\right\}$$

$$\Rightarrow \quad \sin 2\theta_k \cos k \mp \left(\frac{\lambda-1}{\lambda+1}\right) \sin k \cos 2\theta_k = 0$$

$$\Rightarrow \boxed{\tan 2\theta_k = -\left(\frac{1-\lambda}{1+\lambda}\right) \tan k}$$

$$H = \sum_{k>0} \Lambda_k \left[\eta^+(k)\eta(k) + \eta^+(-k)\eta(-k)\right] - a^+(0)a(0) + \text{const.}$$

$$\text{const.} = -2 \sum_{k>0} \left(2 v_k^2 \cos k \mp \frac{\lambda-1}{\lambda+1} \sin k \; 2 u_k v_k\right)$$

$$2 v_k^2 = v_k^2 - u_k^2 + 1 = 1 - (u_k^2 - v_k^2) = 1 - \cos 2\theta_k = 1 - \frac{1}{\sqrt{1 + \left(\frac{1-\lambda}{1+\lambda}\right)^2 \tan^2 k}}$$

$$\Rightarrow \text{const} = -2 \sum_{k>0} \cos k + 2 \sum_{k>0} \left[\cos k\,(u_k^2 - v_k^2) \mp \frac{1-\lambda}{1+\lambda} \sin k \; 2 u_k v_k\right]$$

$$= -2 \overbrace{\sum_{k>0} \cos k}^{N\to\infty \;\; = 0} + 2 \sum_{k>0} \left(\cos k \cos 2\theta_k \mp \left(\frac{1-\lambda}{1+\lambda}\right) \sin k \sin 2\theta_k\right)$$

$$\cancel{\text{const} = 2 \sum_{k>0} \frac{\cos k}{\cos 2\theta_k} = -2 \sum_{k>0} \left[\cos^2 k + \left(\frac{1-\lambda}{1+\lambda}\right)^2 \sin^2 k\right]^{1/2}}$$

$$\cancel{\cos 2\theta} \qquad \text{const} = 2 \sum_{k>0} \cos k \cos 2\theta_k \left[1 \mp \left(\frac{1-\lambda}{1+\lambda}\right) \tan k \tan 2\theta_k\right]$$

$$= -\sum_k \frac{1}{2} \Lambda_k$$

$$\Lambda_k = -2\left[(u_k^2 - v_k^2)\cos k + \left(\frac{1-\lambda}{1+\lambda}\right) 2u_k v_k \sin k\right]$$

$$\Lambda_k = -2\cos k \cos 2\theta_k \left[1 + \left(\frac{1-\lambda}{1+\lambda}\right)\tan k \tan 2\theta_k\right]$$

$$|\Lambda_k| = 2\left[\cos^2 k + \left(\frac{1-\lambda}{1+\lambda}\right)^2 \sin^2 k\right]^{1/2}$$

$$\cos 2\theta_k = \pm \frac{1}{\sqrt{1+\left(\frac{1-\lambda}{1+\lambda}\right)^2 \tan^2 k}}$$

We have to determine the signs. I choose: $\cancel{\mathrm{sgn}\cos 2\theta_k > 0}$

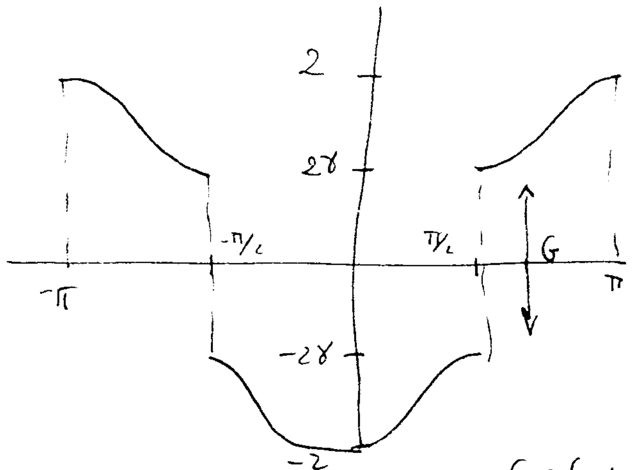
$\cancel{\Rightarrow \mathrm{sgn}\,\Lambda_k = \mathrm{sgn}\cos k}$

$$\mathrm{sgn}\cos 2\theta_k = \mathrm{sgn}\cos k \qquad \text{for} \qquad |k| \le \frac{\pi}{2}$$

$$\mathrm{sgn}\cos 2\theta_k = -\,\mathrm{sgn}\cos k \qquad \text{for} \qquad |k| > \frac{\pi}{2}$$

$$\Rightarrow \quad \Lambda_k = -2\,\mathrm{sgn}(\cos k)\left[\cos^2 k + \gamma^2 \sin^2 k\right]^{1/2} \qquad \gamma = \frac{1-\lambda}{1+\lambda}$$

With this choice we have to fill the Fermi sea : $|k| \le \frac{\pi}{2}$

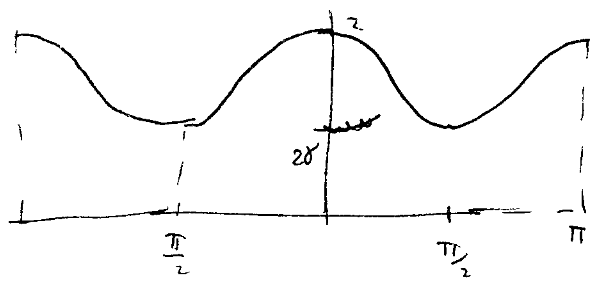


$G = \text{Gap} = 4\gamma$ (for two-particle state)

$$\Rightarrow H = \sum_{k \,(\text{all } k)} \Lambda_k \eta_k^\dagger \eta_k - \frac{1}{2}\sum_k \Lambda_k$$

Filled Fermi sea $\quad |gnd\rangle = \prod_{|k| \le \frac{\pi}{2}} \eta^\dagger(k)\,|0\rangle$

Alternatively we could have chosen the sign of $\Lambda_k > 0$ and we have the particle-hole picture. The ground state is $|0\rangle$ and the dispersion law is



With any choice we see that the ground state energy is the same.

$$|gnd\rangle = \prod_{|k| \le \pi/2} \eta_k^{+} |0\rangle$$

$$H|gnd\rangle = \Big(\sum_{|k| \le \frac{\pi}{2}} \Lambda_k - \frac{1}{2} \sum_{\substack{k \\ (\text{all } k)}} \Lambda_k \Big) |gnd\rangle$$

but $\sum_{|k| \le \frac{\pi}{2}} \Lambda_k = - \sum_{|k| > \frac{\pi}{2}} \Lambda_k < 0 \quad \Rightarrow \sum_{\substack{k \\ (\text{all } k)}} \Lambda_k = 0$

$$\Rightarrow E_{gnd} = \sum_{|k| \le \frac{\pi}{2}} \Lambda_k = - 2 \overset{(1+\lambda)}{\downarrow} \sum_{|k| \le \frac{\pi}{2}} \left[\cos^2 k + \gamma^2 \sin^2 k\right]^{1/2}$$

$$E_{gnd} = - 2N(1+\lambda) \int_{-\pi/2}^{\pi/2} \frac{dk}{2\pi} \left(\cos^2 k + \gamma^2 \sin^2 k\right)^{1/2}$$

$$\mathcal{E}_0(\gamma) = \frac{E_{gnd}}{N} = - \frac{2}{\pi} \int_0^{\pi/2} dk (1+\lambda) \left(1 - (1-\gamma^2) \sin^2 k\right)^{1/2}$$

$$\mathcal{E}_0(\gamma) = - \frac{2(1+\lambda) E\left(\frac{\pi}{2}, \sqrt{1-\gamma^2}\right)}{\pi} \qquad \text{(elliptic function)}$$

As $\lambda \to 1 \Rightarrow \gamma \to 0$

$$\mathcal{E}_0(\lambda) \approx - \frac{2(1+\lambda)}{\pi}\left[1 + \frac{\gamma^2}{4}\left(\ln \frac{16}{\gamma^2} - 1\right) + O(\gamma^4)\right] \sim f(\lambda) \quad \text{(free energy)}$$

$$\Rightarrow \frac{\partial^2 \mathcal{E}}{\partial \lambda^2} \approx \frac{2^{(1+\lambda)}}{\pi} \ln\left(\frac{|1-\lambda|}{2}\right)$$ which diverges logarithmically at $\lambda = 1$ + finite terms.

Thus we find a singularity at $\lambda = 1$

We have observed previously that at $\lambda=1$ the symmetry is bigger that when $\lambda \neq 1$. Indeed the symmetry is continuous. At this very same point the gap is found to vanish

$$G = 2\gamma = 2\left|\lambda - 1\right| \xrightarrow[\lambda \to 1]{} 0$$

At this point a phase transition takes place. For $\lambda > 1$ the ground state is doubly degenerate and the up-down symmetry is broken. For $\lambda < 1$ the ground state is ~~complete~~ non-degenerate and symmetric. At $\lambda=1$ the theory has a critical point. There is an excitation that becomes massless and the correlations are longer ranged.

Summary: We started with the IMTF $(d=1)$

$$H = -\sum_n \sigma_1(n) - \lambda \sum_n \sigma_3(n)\sigma_3(n+1)$$

$\downarrow$ doubling + dual transf. and energy rescaling.

$$H = -(1+\lambda)\sum_n \left[\tau_1(n)\tau_1(n+1) + \tau_3(n)\tau_3(n+1)\right] - \left(1-\lambda\right)\sum_n \left[\tau_3(n)\tau_3(n+1) - \tau_1(n)\tau_1(n+1)\right]$$

$\downarrow$ Jordan-Wigner transf.

$$\psi^+(n) = \left[\prod_{m<n} (-\tau_2(m))\right] \tau^+(n)$$

$$\psi(n) = \left[\prod_{m<n} (-\tau_2(m))\right] \tau^-(n)$$

$\downarrow$

$$H = -\sum_n (1+\lambda)\left[\psi^+(n)\psi(n+1) + \psi^+(n+1)\psi(n)\right] - \left(1-\lambda\right)\sum_n \left[\psi^+(n)\psi^+(n+1) + \psi(n+1)\psi(n)\right]$$

→ Fourier transform + Bogoliubov transformation

$$\psi(n) = \frac{1}{2N+1} \sum_{|s| \le N} e^{-ikn} a(k)$$

$$k = \frac{2\pi s}{2N+1}$$

$$\psi(n) = \int_{-\pi}^{\pi} \frac{dk}{2\pi} e^{-ikn} a(k) = \int_{0}^{\pi} \frac{dk}{2\pi} \Big[e^{-ikn} \left(u_k \eta(k) - i v_k \eta^{+}(-k) \right) +$$

$$+ e^{ikn} \left(u_k \eta(-k) + i v_k \eta^{+}(k) \right) \Big]$$

↓

Hamiltonian:

$$H = \sum_k \Lambda_k \eta^{+}(k) \eta(k) - \frac{1}{2} \sum_k \Lambda_k \qquad \text{all } k.$$

$$\Lambda_k = -2 \,\mathrm{sgn}(\cos k) \sqrt{\cos^2 k + \gamma^2 \sin^2 k} \quad (1+\lambda)$$

$u_k = \cos\theta_k$ $\qquad \gamma = \frac{1-\lambda}{1+\lambda}$

$v_k = \sin\theta_k$

$$\tan 2\theta_k = -\left(\frac{1-\lambda}{1+\lambda}\right) \tan k$$

Green's Functions:

$$G_F(nt, n't') = -i \langle 0 | T\left[\psi(nt)\, \psi(n't') \right] | 0 \rangle$$

In order to simplify the calculation we'll use the picture in which $|0\rangle$ is the vacuum and $\eta(k)|0\rangle = 0$ all k. In this (particle-hole) picture Λ_k is always positive ($\mathrm{sgn}\cos 2\theta_k = -\mathrm{sgn}\cos k$)

$$\Lambda_k = +2\sqrt{\cos^2 k + \gamma^2 \sin^2 k}$$

$$i\frac{\partial}{\partial t}\langle 0|T\,\psi(nt)\,\psi^{+}(n't')\,|0\rangle = i\,\delta(t-t')\,\delta_{nn'} \;+$$

$$\begin{aligned}
&-(1+\lambda)\quad \langle 0|T\,\psi(n+1,t)\;\psi^{+}(n't')\,|0\rangle \\
&-(1+\lambda)\quad \langle 0|T\,\psi(n-1,t)\;\psi^{+}(n't')|0\rangle \\
&-(1-\lambda)\quad \langle 0|\,T\;\psi^{+}(n+1,t)\;\psi^{+}(n't')|0\rangle \\
&+(1-\lambda)\quad \langle 0|\;T\;\;\psi^{+}(n-1,t)\;\psi^{+}(n't')|0\rangle
\end{aligned}$$

Likewise

$$i\frac{\partial}{\partial t}\langle 0|T\;\;\psi^{+}(nt)\;\psi^{+}(n't')|0\rangle = \quad \langle 0|\;\;T\;\;i\frac{\partial\psi^{+}}{\partial t}(nt)\;\;\psi^{+}(n't')\,|0\rangle$$

$$\begin{aligned}
i\frac{\partial}{\partial t}\langle 0|T\psi^{+}(nt)\;\psi^{+}(n't')|0\rangle = \quad &(1+\lambda)\quad \langle 0|T\;\psi^{+}(n+1,t)\;\psi^{+}(n't')|0\rangle \\
&+(1+\lambda)\;\;\langle 0|T\;\psi^{+}(n-1,t)\;\;\psi^{+}(n't')|0\rangle \\
&+(1-\lambda)\quad \langle 0|T\;\psi^{+}(n+1,t)\;\;\psi^{+}(n't')|0\rangle \\
&-(1-\lambda)\quad \langle 0|T\;\;\psi(n-1,t)\quad \psi^{+}(n't')|0\rangle
\end{aligned}$$

Easy to prove that

$$\hat{G}(nt|n't') = \;-i\langle 0|T\;\;\psi^{+}(nt)\;\psi(n't')|0\rangle = -\;G(n't'|nt) = -G^{*}(nt,n't')$$

Same with $\hat{R}$

$$\hat{R}(nt|n't') = \;-i\langle 0|T\;\;\psi(nt)\;\;\psi(n't')|0\rangle = \;-\;\;R(nt|n't')^{*}$$

$$\begin{aligned}
i\;\frac{\partial G}{\partial t}(nt|n't') \;+\; (1+\lambda)\,\big[G(n+1,t|n't') + G(n-1,t|n't')\big] + \\
+(1-\lambda)\,\big[\;R(n+1,t|n't') - R\,(n-1,t|n't')\big] = \quad \delta(t-t')\,\delta_{nn'}
\end{aligned}$$

$$\begin{aligned}
i\;\frac{\partial R}{\partial t}(nt|n't') - (1+\lambda)\,\big[\;R(n+1,t|n't') + \;R(n-1,t|n't')\big] + \\
-\;(1-\lambda)\big[\,G(n+1,t|n't') - G(n-1,t|n't')\big] = \quad 0
\end{aligned}$$

Green's Functions

$$G (nt/n't') = -i \langle 0 | T(\psi(nt)\, \psi^{+}(n't'))|0\rangle$$

$$R \quad (nt/n't') = -i \langle 0 | T(\psi^{+}(nt)\, \psi^{+}(n't'))|0\rangle$$

$$\psi(n,t) = e^{iHt}\, \psi(n,0)\, e^{-iHt}$$

$$\frac{\partial \psi}{\partial t}(n,t) = -i\,[H, \psi(n,t)]$$

$$i\frac{\partial \psi}{\partial t}(n,t) = -(1+\lambda)\{[\psi^{+}(nt)\,\psi(n+1,t), \psi(nt)] + [\psi^{+}(nt)\,\psi(n-1,t), \psi$$

$$-(1-\lambda)\{[\psi^{+}(nt)\,\psi^{+}(n+1,t), \psi(nt)] + [\psi^{+}(n-1,t)\,\psi^{+}(nt), \psi(nt)$$

$$[\psi^{+}(n)\,\psi(n+1), \psi(n)] = -\psi(n+1)$$

$$[\psi^{+}(n)\,\psi(n-1), \psi(n)] = -\psi(n-1)$$

$$[\psi^{+}(n)\,\psi^{+}(n+1), \psi(n)] = -\psi^{+}(n+1)$$

$$[\psi^{+}(n-1)\,\psi^{+}(n), \psi(n)] = +\psi^{+}(n-1)$$

$$i\frac{\partial \psi}{\partial t}(n,t) = -(1+\lambda)(\psi(n+1,t) + \psi(n-1,t)) - (1-\lambda)(\psi^{+}(n+1,t) - \psi^{+}(n-1,t)$$

$$i\frac{\partial \psi^{+}}{\partial t}(n,t) = +(1+\lambda)(\psi^{+}(n+1,t) + \psi^{+}(n-1,t)) + (1-\lambda)(\psi(n+1,t) - \psi(n-1,t)$$

$$i\frac{\partial}{\partial t}\langle 0|T\,\psi(nt)\,\psi^{+}(n't')|0\rangle = i\,\delta(t-t')\,\delta_{nn'} + \langle 0|T\, i\frac{\partial \psi}{\partial t}(nt)\,\psi^{+}(n't')|$$

$x_1 = na$ $\qquad \Lambda \approx \frac{\pi}{2a}$

$(n-n') \gg 1$

$$S(x_1-x_1', t-t') \approx \int_{-\infty}^{+\infty} \frac{dq_0}{2\pi} \int_{-\Lambda}^{\Lambda} \frac{dq_1}{2\pi} \frac{e^{i[q_0(x_0-x_0') - q_1(x_1-x_1')]}}{q_0^2 - q_1^2 - m^2 + i\varepsilon} \times$$

$$\times \quad i^{n-n'} \left(1+(-1)^{n-n'}\right)$$

$$m^2 = \frac{(1-\lambda)^2}{4\lambda a^2} = \text{"mass" or gap.}$$

$$S(x_1-x_1', t-t') \approx \frac{1}{2\pi} K_0\left(m\sqrt{x^2+\frac{1}{\Lambda^2}}\right) \times i^{n-n'}\left(1+(-1)^{n-n'}\right)$$

in imaginary time

$$x^2 = (x_1-x_1')^2 + (x_0-x_0')^2$$

$\uparrow$ imaginary

At $\lambda=1$ $\qquad m=0$

$$S(x_1-x_1', x_0-x_0') \approx -\frac{1}{4\pi} \ln\left[c\mu^2\left(x^2+\frac{1}{\Lambda^2}\right)\right]$$

μ: dimension of $\frac{1}{a}$

$$\Rightarrow G \sim i\frac{\partial S}{\partial t} \approx \frac{1}{4\pi} \frac{2x_0}{x_1^2 - x_0^2} = \frac{1}{2\pi} \frac{x_0}{x_1^2-x_0^2}$$

at large separations

Equal-time green's function

$$G(n,n') = \langle 0|\psi(n)\;\psi^\dagger(n')|0\rangle = \langle 0|T(\psi(n,t)\;\psi^\dagger(n',t'))|0\rangle \quad t'\to t^+$$

$$= \int_0^{\pi}\frac{dk}{2\pi}\int_0^{\pi}\frac{dk'}{2\pi}\Big[\langle 0|\eta(k)\,\eta(k')^\dagger|0\rangle\; u_k u_{k'}\; \cancel{e^{-ik(n-n')}}\, e^{-i(kn-k'n')}$$

$$+\; u_k u_{k'}\;\langle 0|\eta(k)\eta(-k')^\dagger|0\rangle\;\cancel{e^{ik(n-n')}}\, e^{i(kn-k'n')}\Big]$$

$$G(n-n') = \int_0^{\pi}\frac{dk}{(2\pi)^2}\;2u_k^2\cos k(n-n')$$

$$2u_k^2 = 1+\cos 2\theta_k$$

$$\Rightarrow\; G(n-n') = \int_0^{\pi}\frac{dk}{(2\pi)^2}\;\cos 2\theta_k\,\cos k(n-n') + \frac{1}{2\pi}\,\delta_{n,n'}$$

$$G(n-n') = \int_0^{\pi}\frac{dk}{(2\pi)^2}\;\frac{\mathrm{sgn}(\cos k)\,\cos k(n-n')}{\sqrt{1+\gamma^2\tan^2 k}} + \frac{1}{2\pi}\,\delta_{nn'}$$

$$G(n-n') = -\int_0^{\pi}\frac{dk}{(2\pi)^2}\;\frac{\cos k\,\cos k(n-n')}{\sqrt{\cos^2 k+\gamma^2\sin^2 k}} + \frac{1}{2\pi}\,\delta_{nn'}$$

$$G(n,n') = -\frac{1}{2}\int_0^{\pi}\frac{dk}{(2\pi)^2}\;\frac{\cos(k(n-n'+1))+\cos(k(n-n'-1))}{\sqrt{\cos^2 k+\gamma^2\sin^2 k}} + \frac{\delta_{nn'}}{2\pi}$$

$|n-n'|\to\infty$

$$G(n) \approx -\int_0^{\pi}\frac{dk}{(2\pi)^2}\;\frac{\cos kn}{(\cos^2 k+\gamma^2\sin^2 k)^{1/2}} + \frac{\delta_{n,0}}{2\pi}$$

most of the contribution comes from $k \approx \frac{\pi}{2}$ as $n\to\infty$ ($\gamma\to$

$k = q + \frac{\pi}{2}$

$$G(n) \approx - \int_{-\pi/2}^{\pi/2} \frac{dq}{(2\pi)^2} \frac{\cos\left(q + \frac{\pi}{2}\right) n}{\sqrt{\sin^2 q + \gamma^2 \cos^2 q}} + \frac{1}{2\pi} \delta_{n,0}$$

$$G(n) \approx - \int_{-\pi/2}^{\pi/2} \frac{dq}{(2\pi)^2} \operatorname{Re} \frac{e^{iqn} e^{i\pi n/2}}{\sqrt{q^2 + \gamma^2}} \underset{n \to \infty}{\approx} - \frac{e^{-\gamma n}}{\sqrt{\gamma n}} \frac{\cos \pi n/2}{\sqrt{8\pi^2}}$$

Thus the Green's function decays exponentially with a rate given by $\frac{1}{\gamma}$

$$\xi = \frac{1}{\gamma} \approx \frac{2}{|1-\lambda|} \underset{\lambda \to 1}{\longrightarrow} \infty$$

At $\gamma = 0$

$$G(n) \simeq - \int_{-\pi/2}^{\pi/2} \frac{dq}{(2\pi)^2} \frac{\cos\left(q + \frac{\pi}{2}\right) n}{\sqrt{\sin^2 q + \gamma^2 \cos^2 q}} = - \frac{(-1)^{\frac{n+1}{2}}}{2\pi^2 n}$$

Power law falloff

(n odd)