Chapter 1

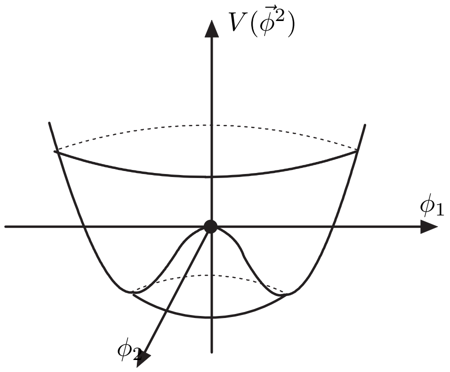
Perturbation Theory and Feynman Graphs

1.1 Perturbation Theory

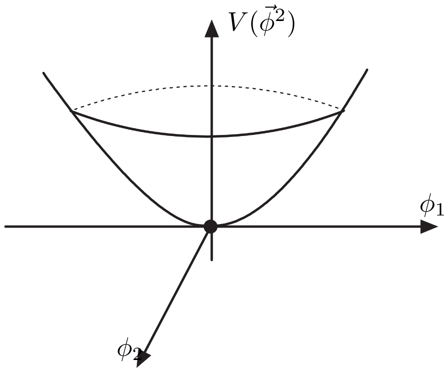
We have seen how to evaluate functional integrals for free fields, i.e. gaussian models. In general, all interesting systems may (or will) contain interactions. These interactions may appear explicitly as a result of constraints. A classic example is the $\lambda\phi^4$ theory:

$$\mathcal{L} = \frac{1}{2}(\nabla\vec{\phi})^2 + \frac{m_0^2}{2}\vec{\phi}^2 + \frac{\lambda}{4!}(\vec{\phi}^2)^2 \tag{1.1}$$

For $m_0^2 < 0$ the potential looks like :



while for $m_0^2 > 0$:



We can re-parametrize $\mathcal{L}$ as follows :

$$\mathcal{L}(\phi) = \frac{1}{2}(\nabla\vec{\phi})^2 + \frac{\lambda}{4!}(\vec{\phi}^2 - f^2)^2 \tag{1.2}$$

with $2\frac{\lambda}{4!}f^2 \equiv -\frac{m_0^2}{2}$.

If the potential is infinitely deep ($\lambda \to \infty$) then the system is effectively constrained: $\vec{\phi}^2 = f^2$ (*hard spin*). Then a rescaling $\vec{\phi} = f\vec{n}$ yields:

$$\mathcal{L} = \frac{f^2}{2}(\nabla\vec{n})^2 \tag{1.3}$$

with the constraint $\vec{n}^2 = 1$. Then, the generating functional becomes:

$$\mathcal{Z} = \int \mathcal{D}\vec{n} e^{-\int d^dx \frac{1}{2\tau}(\nabla\vec{n})^2} \delta(\vec{n}^2 - 1) \qquad f^2 \equiv \frac{1}{\tau} \tag{1.4}$$

This model, which is known as the *non-linear sigma model*, is in fact a continuum approximation to the Heisenberg model of a ferromagnet:

$$\mathcal{H}_{Lattice} = \frac{1}{\tau} \sum_{<\vec{r},\vec{r}'>} \vec{S}(\vec{r}) \cdot \vec{S}(\vec{r}') \qquad \vec{S}^2 = 1 \tag{1.5}$$

It also appears in High Energy Physics in the description of phenomena such as chiral symmetry breaking, pion physics etc. Interactions may arise from the coupling of two fields otherwise free. For example, in QED:

$$\mathcal{L}_{QED} = \bar{\psi}(\imath\gamma_\mu\partial^\mu - m)\psi - e\bar{\psi}\gamma_\mu\psi A^\mu - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} \tag{1.6}$$

I will describe here a technique to generate in a systematic fashion an expansion of physical quantities in powers of a coupling constant. For example, in a ϕ^4 theory with $m_0^2 > 0$ (N=1) :

$$\mathcal{L} = \frac{1}{2}(\nabla\phi)^2 + \frac{m_0^2}{2}\phi^2 + \frac{\lambda}{4!}\phi^4 \tag{1.7}$$

The ground-state configuration minimizes the Lagrangian $\delta\mathcal{L} = 0$:

$$-\nabla^2\phi + m_0^2\phi + \tfrac{\lambda}{3!}\phi^3 = 0 \tag{1.8}$$
$$\phi = const = 0 \qquad \text{for } m_0^2 > 0 \tag{1.9}$$

We will expand around this ground state. This happens to coincide with an expansion in powers of λ :

$$\mathcal{Z}[J] = \mathcal{N}\int \mathcal{D}\phi e^{-\int d^dx\mathcal{L}+\int d^dxJ\phi} \qquad \mathcal{N}^{-1} = \mathcal{Z}[0] \tag{1.10}$$

$$\begin{aligned} e^{-\int d^dx\mathcal{L}} &= e^{-\int d^dx[\mathcal{L}_0+\frac{\lambda}{4!}\phi^4]} & (1.11)\\ &= \sum_{n=0}^{\infty}\frac{(-1)^n}{n!}\left[\int d^dx\frac{\lambda}{4!}\phi^4(\mathbf{x})\right]^n e^{-\int d^dx\mathcal{L}_0} & (1.12)\end{aligned}$$

where $\mathcal{L}_0 = \frac{1}{2}(\nabla\phi)^2 + \frac{m_0^2}{2}\phi^2$. Expanding the interacting term, we take:

$$\begin{aligned} \mathcal{Z}[J] &= \\ &= \mathcal{N}\sum_{n=0}^{\infty}\frac{(-1)^n}{n!}\int\mathcal{D}\phi e^{-\int d^dx(\mathcal{L}_0-J\phi)}\left[\int d^dx\frac{\lambda}{4!}\phi^4(\mathbf{x})\right]^n & (1.13)\\ &= \mathcal{N}\sum_{n=0}^{\infty}\frac{1}{n!}\left(-\frac{\lambda}{4!}\right)^n\int\mathcal{D}\phi e^{-S_0[J,\phi]} \\ &\qquad \times\int d^dx_1\ldots\int d^dx_n\phi^4(x_1)\ldots\phi^4(x_n) & (1.14)\\ &= \mathcal{N}\left(\int\mathcal{D}\phi e^{-S_0[J,\phi]}\right)\sum_{n=0}^{\infty}\frac{1}{n!}\left(-\frac{\lambda}{4!}\right)^n \\ &\qquad \times\int d^dx_1\ldots\int d^dx_n\langle\phi^4(x_1)\ldots\phi^4(x_n)\rangle_J & (1.15)\end{aligned}$$

where

$$\langle A(\phi)\rangle_J = \frac{\int\mathcal{D}\phi A(\phi)e^{-S_0[J,\phi]}}{\int\mathcal{D}\phi e^{-S_0[J,\phi]}} \tag{1.16}$$

On the other hand, an operator in the expansion (e.g. $\phi^4(x_1)e^{-S_0[\phi,J]}$) can be obtained as follows:

$$\phi^4(x_1)e^{-S_0[\phi,J]} = \frac{\delta^4}{\delta J^4(x_1)}e^{-S_0[\phi,J]} \tag{1.17}$$

Thus, we may write:

$$\mathcal{Z}[J] = \sum_{n=0}^{\infty} \left(-\frac{\lambda}{4!}\right)^n \frac{1}{n!} \int \mathcal{D}\phi \int d^d x_1 \frac{\delta^4}{\delta J^4(x_1)} \cdots \int d^d x_n \frac{\delta^4}{\delta J^4(x_n)} e^{-S_0[\phi,J]} \tag{1.18}$$

By re-exponentiating we get (formally) :

$$\mathcal{Z}[J] = \mathcal{N} e^{-\frac{\lambda}{4!}\int d^d x \frac{\delta^4}{\delta J(x)^4}} \int \mathcal{D}\phi e^{-S_0[\phi,J]} \tag{1.19}$$

$$\mathcal{Z}[J] = \mathcal{N} e^{-\int d^d x \mathcal{L}_{int}\left[\frac{\delta}{\delta J(x)}\right]} \int \mathcal{D}\phi e^{-S_0[\phi,J]} \tag{1.20}$$

where $\mathcal{L}_{int}[\phi] = \frac{\lambda}{4!}\phi^4(x)$.

However,

$$\int \mathcal{D}\phi e^{-S_0[\phi,J]} = \mathcal{Z}_0[J] = \mathcal{Z}_0[0] e^{\frac{1}{2}\int d^d x_1 \int d^d x_2 J(x_1) G_0(x_1,x_2) J(x_2)} \tag{1.21}$$

So,

$$\mathcal{Z}[J] = \mathcal{N}\ e^{-\int d^d x \mathcal{L}_{int}\left[\frac{\delta}{\delta J(x)}\right]} e^{\frac{1}{2}\int d^d x_1 \int d^d x_2 J(x_1) G_0(x_1,x_2) J(x_2)} \tag{1.22}$$

The constant $\mathcal{N}$ may be calculated from the normalization condition $\mathcal{Z}[0] = 1$:

$$\mathcal{N}^{-1} = e^{-\int d^d x \mathcal{L}_{int}\left[\frac{\delta}{\delta J(x)}\right]} e^{\frac{1}{2}\int d^d x_1 \int d^d x_2 J(x_1) G_0(x_1,x_2) J(x_2)} \tag{1.23}$$

I will use the formula (1.22)to derive an expansion for $G_2(x_1, x_2)$:

$$G_2(x,y) = \langle \phi(x)\phi(y) \rangle = \left. \frac{\delta^2 \mathcal{Z}[J]}{\delta J(x) \delta J(y)} \right|_{J=0} \tag{1.24}$$

$$G_2(x,y) = \mathcal{N} \sum_{n=0}^{\infty} \left(-\frac{\lambda}{4!}\right)^n \Bigg[\int d^d x_1 \ldots \int d^d x_n \frac{\delta^2}{\delta J(x)\delta J(y)} \frac{\delta^4}{\delta J^4(x_1)} \cdots \frac{\delta^4}{\delta J^4(x_n)} \mathcal{Z}_0[J] \Bigg|_{J=0} \Bigg] \tag{1.25}$$

1. $n = 0$

$$G_2(x,y) = \mathcal{N} \frac{\delta^2}{\delta J(x)\delta J(y)} \mathcal{Z}_0[J]|_{J=0} = G_0(x,y) = \tag{1.26}$$

   where $N^{-1} = 1$.

2. $n = 1$

$$G_2^{(1)}(x,y) = G_0(x,y) + \delta G_2^{(1)}(x,y) \tag{1.27}$$

$$\delta G_2^{(1)}(x,y) = \left(-\frac{\lambda}{4!}\right)^4 \frac{N_1}{1!} \frac{\delta^2}{\delta J(x)\delta J(y)} \Big[\int d^d x_1 \frac{\delta^4}{\delta J^4(x_1)} e^{\frac{1}{2}\int d^d y_1 \int d^d y_2 J(y_1) G_0(y_1,y_2) J(y_2)} \Big|_{J=0} \Big] \tag{1.28}$$

$$N_1^{-1} = 1 + \left(\frac{-\lambda}{4!}\right)^1 \frac{1}{1!} \times \int d^d x_1 \frac{\delta^4}{\delta J^4(x_1)} e^{\frac{1}{2}\int d^d y_1 \int d^d y_2 J(y_1) G_0(y_1,y_2) J(y_2)} \Big|_{J=0} \tag{1.29}$$

Now,

$$\frac{\delta}{\delta J(x_1)} e^{\frac{1}{2}\int d^d y_1 \int d^d y_2 J(y_1) G_0(y_1,y_2) J(y_2)} = \frac{1}{2}\Big[\int d^d y_1 J(y_1) G_0(y_1,x_1) + + \int d^d y_2 G_0(x_1,y_2) J(y_2) \Big] e^{\frac{1}{2}\int d^d y_1 \int d^d y_2 J(y_1) G_0(y_1,y_2) J(y_2)} \tag{1.30}$$

$$\frac{\delta^2}{\delta J(x_1)^2} e^{\frac{1}{2}\int d^d y_1 \int d^d y_2 J(y_1) G_0(y_1,y_2) J(y_2)} = \frac{1}{2}\left(G_0(x_1,x_1) + G_0(x_1,x_1)\right) \times e^{\frac{1}{2}\int d^d y_1 \int d^d y_2 J(y_1) G_0(y_1,y_2) J(y_2)} + \Big\{ \frac{1}{2}\Big[\int d^d y_1 J(y_1) G_0(y_1,x_1) + + \int d^d y_2 G_0(x_1,y_2) J(y_2) \Big] \Big\}^2 e^{\frac{1}{2}\int d^d y_1 \int d^d y_2 J(y_1) G_0(y_1,y_2) J(y_2)} \tag{1.31}$$

$$\frac{\delta^3}{\delta J(x_1)^3} e^{\frac{1}{2}\int d^d y_1 \int d^d y_2 J(y_1) G_0(y_1,y_2) J(y_2)} = G_0(x_1,x_1)\frac{1}{2}\Big[\int d^d y_1 \big[J(y_1) \times G_0(y_1,x_1)\big] + \int d^d y_2 G_0(x_1,y_2) J(y_2) \Big] e^{\frac{1}{2}\int d^d y_1 \int d^d y_2 J(y_1) G_0(y_1,y_2) J(y_2)} + + \left\{ \frac{1}{2}\left[\int d^d y_1 J(y_1) G_0(y_1,x_1) + \int d^d y_2 G_0(x_1,y_2) J(y_2) \right] \right\}^3 \times e^{\frac{1}{2}\int d^d y_1 \int d^d y_2 J(y_1) G_0(y_1,y_2) J(y_2)} + +2\frac{1}{2}\left[\int d^d y_1 J(y_1) G_0(y_1,x_1) + \int d^d y_2 G_0(x_1,y_2) J(y_2) \right] \times \frac{1}{2}\left(G_0(x_1,x_1) + G_0(x_1,x_1)\right) e^{\frac{1}{2}\int d^d y_1 \int d^d y_2 J(y_1) G_0(y_1,y_2) J(y_2)} \tag{1.32}$$

$$= 3G_0(x_1,x_1)\frac{1}{2}\left[\int d^dy_1 J(y_1)G_0(y_1,x_1) + \int d^dy_2 G_0(x_1,y_2)J(y_2)\right] \times e^{\frac{1}{2}\int d^dy_1\int d^dy_2 J(y_1)G_0(y_1,y_2)J(y_2)}$$
$$+\left\{\frac{1}{2}\left[\int d^dy_1 J(y_1)G_0(y_1,x_1) + \int d^dy_2 G_0(x_1,y_2)J(y_2)\right]\right\}^3 \times e^{\frac{1}{2}\int d^dy_1\int d^dy_2 J(y_1)G_0(y_1,y_2)J(y_2)} \quad (1.33)$$

$$\frac{\delta^4}{\delta J(x_1)^4}e^{\frac{1}{2}\int d^dy_1\int d^dy_2 J(y_1)G_0(y_1,y_2)J(y_2)} = 3(G_0(x_1,x_1))^2 \times e^{\frac{1}{2}\int d^dy_1\int d^dy_2 J(y_1)G_0(y_1,y_2)J(y_2)}$$
$$+3G_0(x_1,x_1)\left\{\frac{1}{2}\left[\int d^dy_1 J(y_1)G_0(y_1,x_1) + \int d^dy_2 G_0(x_1,y_2)J(y_2)\right]\right\}^2 \times e^{\frac{1}{2}\int d^dy_1\int d^dy_2 J(y_1)G_0(y_1,y_2)J(y_2)}$$
$$+3\left\{\frac{1}{2}\left[\int d^dy_1 J(y_1)G_0(y_1,x_1) + \int d^dy_2 G_0(x_1,y_2)J(y_2)\right]\right\}^2 \times\frac{1}{2}(G_0(x_1,x_1)+G_0(x_1,x_1))e^{\frac{1}{2}\int d^dy_1\int d^dy_2 J(y_1)G_0(y_1,y_2)J(y_2)}$$
$$+\left\{\frac{1}{2}\left[\int d^dy_1 J(y_1)G_0(y_1,x_1) + \int d^dy_2 G_0(x_1,y_2)J(y_2)\right]\right\}^4 \times e^{\frac{1}{2}\int d^dy_1\int d^dy_2 J(y_1)G_0(y_1,y_2)J(y_2)} \quad (1.34)$$

$$\frac{\delta^4}{\delta J(x_1)^4}e^{\frac{1}{2}\int d^dy_1\int d^dy_2 J(y_1)G_0(y_1,y_2)J(y_2)}|_{J=0} = 3(G_0(x_1,x_1))^2 \quad (1.35)$$

$$\frac{\delta}{\delta J(x)}\frac{\delta}{\delta J(x')}\frac{\delta^4}{\delta J(x_1)^4}e^{\frac{1}{2}\int d^dy_1\int d^dy_2 J(y_1)G_0(y_1,y_2)J(y_2)}|_{J=0} =$$
$$= 3(G_0(x_1,x_1))^2\frac{1}{2}\left(G_0(x,x')+G_0(x',x)\right)$$
$$+3(G_0(x_1,x_1))2\frac{1}{2}\left(G_0(x,x_1)+G_0(x_1,x)\right)\frac{1}{2}\left(G_0(x',x_1)+G_0(x_1,x')\right)$$
$$+3\cdot 2\frac{1}{2}\left(G_0(x_1,x)+G_0(x,x_1)\right)\frac{1}{2}\left(G_0(x_1',x_1)+G_0(x_1,x_1')\right)G_0(x_1,x_1) \quad (1.36)$$

Since $G_0(x,x') = G_0(x',x)$ we have:

$$\frac{\delta}{\delta J(x)}\frac{\delta}{\delta J(x')}\frac{\delta^4}{\delta J(x_1)^4}e^{\frac{1}{2}\int d^dy_1\int d^dy_2 J(y_1)G_0(y_1,y_2)J(y_2)}|_{J=0} =$$
$$= 3(G_0(x_1,x_1))^2G_0(x,x') + 12G_0(x_1,x_1)G_0(x,x_1)G_0(x,x') \quad (1.37)$$

Note: Equivalently we can derive this result by noticing that:

$$\frac{\delta}{\delta J(x)}\frac{\delta}{\delta J(x')}\frac{\delta^4}{\delta J(x_1)^4}e^{\frac{1}{2}\int d^dy_1\int d^dy_2 J(y_1)G_0(y_1,y_2)J(y_2)}|_{J=0} =$$

$$= \frac{\delta}{\delta J(x)}\frac{\delta}{\delta J(x')}\frac{\delta^4}{\delta J(x_1)^4}\frac{1}{3!}\left[\frac{1}{2}\int d^dy_1\int d^dy_2 J(y_1)G_0(y_1,y_2)J(y_2)\right]^3|_{J=0} \quad (1.38)$$

since all other terms are necessarily zero.
This amounts to an expression in which pairs of points, say x and x' *or* x with x_1 *or* x' with x_1 *or* x_1 with itself, are connected in all possible ways and each connection (or *contraction*) is represented by a propagator factor.

$$\delta G_2^{(1)}(x,y) = \mathcal{N}\left(-\frac{\lambda}{4!}\right)^1\frac{1}{1!}\int d^dx_1\{12G_0(x,x_1)G_0(x_1,y)G_0(x_1,x_1) + 3[G_0(x_1,x_1)]^2G_0(x,y)\} \quad (1.39)$$

So,

$$G_2^{(1)}(x,y) = \mathcal{N}\left[G_0(x,y) + \left(-\frac{\lambda}{4!}\right)^1\frac{1}{1!}\int d^dx_1\Big\{12G_0(x,x_1)G_0(x_1,y) \times G_0(x_1,x_1) + 3[G_0(x_1,x_1)]^2G_0(x,y)\Big\}\right] \quad (1.40)$$

The normalization constant $\mathcal{N}$ up to terms $O(\lambda)$ is:

$$\mathcal{N}^{-1} = 1 + \left(-\frac{\lambda}{4!}\right)^1\frac{1}{1!}\int d^dx_1\left\{3[G_0(x_1,x_1)]^2\right\} \quad (1.41)$$

or

$$\mathcal{N}G_0(x,y) = G_0(x,y) - G_0(x,y)\left(-\frac{\lambda}{4!}\right)^1\frac{1}{1!}\int d^dx_1\left\{3[G_0(x_1,x_1)]^2\right\} + O(\lambda^2) \quad (1.42)$$

By using the above relation, (1.40) can be written as follows :

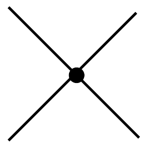
$$G_2^{(1)}(x,y) = G_0(x,y) - G_0(x,y)\left(-\frac{\lambda}{4!}\right)\int d^dx_1 3[G_0(x_1,x_1)]^2 + \left(-\frac{\lambda}{4!}\right)\int d^dx_1 12G_0(x,x_1)G_0(x_1,y)G_0(x_1,x_1) + \left(-\frac{\lambda}{4!}\right)\int d^dx_1 3[G_0(x_1,x_1)]^2G_0(x,y) + O(\lambda^2) \quad (1.43)$$

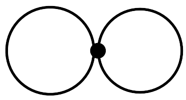
$$G_2^{(1)}(x,y) = G_0(x,y) + \left(-\frac{\lambda}{4!}\right)12\int d^dx_1 G_0(x,x_1)G_0(x_1,y)G_0(x_1,x_1) + O(\lambda^2) \quad (1.44)$$

Example:

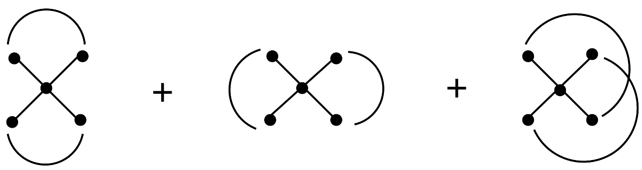
1. *Vacuum Graphs*:

$n = 0$: 1

$n = 1$:  $\rightarrow \left(-\frac{\lambda}{4!}\right)^1 \frac{1}{1!}$

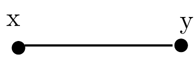
Form the contractions:  $\rightarrow \int d^d x [G_0(x,x)]^2 \left(-\frac{\lambda}{4!}\right) \frac{1}{1!} \overbrace{S}^{\text{symmetry factor}}$

where S=3, since the possible number of contractions are three:

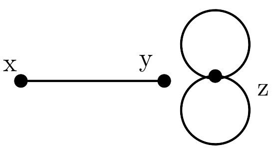


and the normalization constant is $\mathcal{N}^{-1} = 1 + \left(-\frac{\lambda}{4!}\right)^1 \frac{1}{1!} \times 3 \int d^d z [G_0(z,z)]^2$.

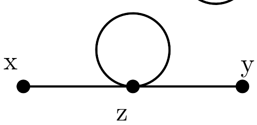
2. *2-point Correlation Function*:

$n = 0$ 

$\equiv \mathcal{N} \times G_0(x,y)$

$n = 1$ 

$\equiv \mathcal{N} \times 3 \left(\frac{-\lambda}{4!}\right)^1 \frac{1}{1!} \int d^d z [G_0(z,z)]^2 G_0(x,y)$



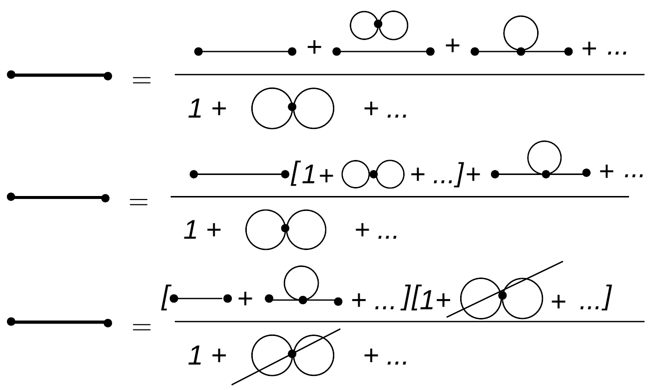
$\equiv \mathcal{N} \times \underbrace{S}_{=4\times 3} \left(\frac{-\lambda}{4!}\right)^1 \frac{1}{1!} \int d^d z G_0(x,z) G_0(z,y) G_0(z,z)$

The 2-point function to order $O(\lambda)$ can be written as follows:

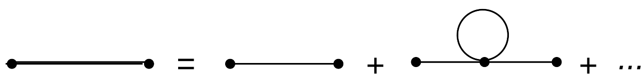
$$\begin{aligned} G_2(x,y) &= \mathcal{N}\Big\{G_0(x,y) + \left(-\frac{\lambda}{4!}\right)3\int d^dz[G_0(z,z)]^2G_0(x,y) \\ &\quad + \left(-\frac{\lambda}{4!}\right)4\times 3\int d^dzG_0(x,z)G_0(z,y)G_0(z,z)\Big\} + O(\lambda^2) \quad (1.45) \\ G_2(x,y) &= \left[1+N_1+O(\lambda^2)\right]\left[G_0(x,y)+G_2^{(1)}(x,y)+O(\lambda^2)\right] \quad (1.46) \\ G_2(x,y) &\cong G_0(x,y)+G_2^{(1)}(x,y)+N_1G_0(x,y)+O(\lambda^2) \quad (1.47) \end{aligned}$$

where $N_1 = -\left(-\frac{\lambda}{4!}\right)3\int d^dz\,[G_0(z,z)]^2$.
So,

$$G_2(x,y) = G_0(x,y) + \left(-\frac{\lambda}{4!}\right)4\times 3\int d^dzG_0(x,z)G_0(z,y)G_0(z,z)$$
$$+\left(-\frac{\lambda}{4!}\right)3\int d^dz[G_0(z,z)]^2G_0(x,y) - \left(-\frac{\lambda}{4!}\right)3\int d^dz[G_0(z,z)]^2G_0(x,y)$$
$$+O(\lambda^2) \quad (1.48)$$



Thus, the *disconnected* graphs cancel against the denominator and finally:



$$G_2(x,y) = G_0(x,y) + \left(-\frac{\lambda}{4!}\right)4\times 3\int d^dzG_0(x,z)G_0(z,y)G_0(z,z) + O(\lambda^2) \quad (1.49)$$

1.2 Cancellation of vacuum graphs

The cancellation found in first order is not accidental. It happens to all orders. Generally we have:

$$\langle\phi(x_1)\phi(x_2)\rangle = \frac{\int \mathcal{D}\phi\phi(x_1)\phi(x_2)e^{-\int d^dx\mathcal{L}_0(\phi(x))-\int d^dx\mathcal{L}_{int}(\phi(x))}}{\int \mathcal{D}\phi e^{-\int d^dx\mathcal{L}_0(\phi(x))-\int d^dx\mathcal{L}_{int}(\phi(x))}} \quad (1.50)$$

$$= \frac{\sum_{p=0}^{\infty}\frac{(-1)^p}{p!}\int \mathcal{D}\phi e^{-\int d^dx\mathcal{L}_0(\phi(x))}\phi(x_1)\phi(x_2)\left[\int d^dx\mathcal{L}_{int}(\phi(x))\right]^p}{\sum_{p=0}^{\infty}\frac{(-1)^p}{p!}\int \mathcal{D}\phi e^{-\int d^dx\mathcal{L}_0(\phi(x))}\left[\int d^dx\mathcal{L}_{int}(\phi(x))\right]^p} \quad (1.51)$$

$$= \frac{\sum_{p=0}^{\infty}\frac{(-1)^p}{p!}\langle\phi(x_1)\phi(x_2)\left[\int d^dx\mathcal{L}_{int}(\phi(x))\right]^p\rangle_0\int \mathcal{D}\phi e^{-\int d^dx\mathcal{L}_0(\phi(x))}}{\sum_{p=0}^{\infty}\frac{(-1)^p}{p!}\langle\left[\int d^dx\mathcal{L}_{int}(\phi(x))\right]^p\rangle_0\int \mathcal{D}\phi e^{-\int d^dx\mathcal{L}_0(\phi(x))}} \quad (1.52)$$

$$= \frac{\sum_{p=0}^{\infty}\frac{(-1)^p}{p!}\int dy_1\dots dy_p\langle\phi(x_1)\phi(x_2)\mathcal{L}_{int}(\phi(y_1))\dots\mathcal{L}_{int}(\phi(y_p))\rangle_0}{\sum_{p=0}^{\infty}\frac{(-1)^p}{p!}\int dy_1\dots dy_p\langle\mathcal{L}_{int}(\phi(y_1))\dots\mathcal{L}_{int}(\phi(y_p))\rangle_0} \quad (1.53)$$

In general,

$$\langle\phi(x_1)\phi(x_2)\mathcal{L}_{int}(\phi(y_1))\dots\mathcal{L}_{int}(\phi(y_p))\rangle_0 =$$
$$= \sum_{k=0}^{p}\binom{p}{k}\langle\phi(x_1)\phi(x_2)\mathcal{L}_{int}(\phi(y_1))\dots\mathcal{L}_{int}(\phi(y_k))\rangle_0^{linked}$$
$$\times\langle\mathcal{L}_{int}(\phi(y_{k+1}))\dots\mathcal{L}_{int}(\phi(y_p))\rangle_0^{vac.graphs} \quad (1.54)$$

So,

$$\langle\phi(x_1)\phi(x_2)\rangle = \frac{1}{\sum_{p=0}^{\infty}\frac{(-1)^p}{p!}\int dy_1\dots dy_p\langle\mathcal{L}_{int}(\phi(y_1))\dots\mathcal{L}_{int}(\phi(y_p))\rangle_0}$$
$$\times\sum_{p=0}^{\infty}\frac{(-1)^p}{p!}\sum_{k=0}^{p}\binom{p}{k}\int dy_1\dots dy_p\langle\mathcal{L}_{int}(\phi(y_{k+1}))\dots\mathcal{L}_{int}(\phi(y_p))\rangle_0^{vac.graphs}$$
$$\times\langle\phi(x_1)\phi(x_2)\mathcal{L}_{int}(\phi(y_1))\dots\mathcal{L}_{int}(\phi(y_k))\rangle_0^{linked} \quad (1.55)$$

$$= \frac{1}{\sum_{p=0}^{\infty}\frac{(-1)^p}{p!}\int dy_1\dots dy_p\langle\mathcal{L}_{int}(\phi(y_1))\dots\mathcal{L}_{int}(\phi(y_p))\rangle_0}$$
$$\times\sum_{k=0}^{\infty}\frac{(-1)^k}{k!}\int dy_1\dots dy_k\langle\phi(x_1)\phi(x_2)\mathcal{L}_{int}(\phi(y_1))\dots\mathcal{L}_{int}(\phi(y_k))\rangle_0^{linked}$$
$$\times\sum_{n=0}^{\infty}\frac{(-1)^n}{n!}\int dy_1\dots dy_n\langle\mathcal{L}_{int}(\phi(y_1))\dots\mathcal{L}_{int}(\phi(y_n))\rangle_0 \quad (1.56)$$

Thus, the following formula generally holds:

$$\langle\phi(x_1)\dots\phi(x_N)\rangle =$$
$$= \sum_{k=0}^{\infty}\frac{(-1)^k}{k!}\int dy_1\dots dy_k\langle\phi(x_1)\dots\phi(x_N)\mathcal{L}_{int}(\phi(y_1))\dots\mathcal{L}_{int}(\phi(y_k))\rangle_0^{linked} \quad (1.57)$$

Rule(Wick's Theorem):

After all derivatives are done we must set $J = 0$. Obviously the only terms to survive are those with no J derivatives. Enery derivative brings down one factor of J and a propagator factor. Thus another derivative will be needed to cancel the factors of J. Thus all derivatives come in pairs (contractions) in all possible ways and for each pair one assigns a propagator factor.

1.3 Feynman Graphs

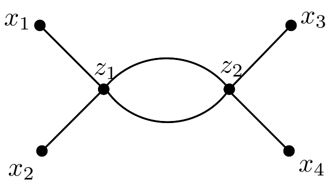
For computing $G_2(x, y)$ at n-th order in perturbation theory we must proceed as follows:

1.3.1 Feynman Rules In Configuration Space

A graph consists of N external points and n internal vertices and must be drawn according to the following rules:

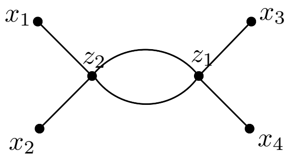
1. Assign a factor $\frac{1}{n!}\left(-\frac{\lambda}{4!}\right)^n$.
2. Assign to every contraction a line (propagator) $G_0(x_1, x_2)$.
3. Using these lines draw all possible contractions between the internal vertices themselves and between them and the external points x,y (or legs).
4. Integrate over all the coordinates of the internal vertices.
5. Rule (3) can be simplified by drawing graphs of different topology proided one multiplies each graph by its multiplicity S (*symmetry factor*).
6. Multiply the above contribution by $\mathcal{N}$(*the vacuum graphs*) calculated up to the same order.

Example: 2^{nd} order graph for the 4-point function.



$$\equiv \left(-\frac{\lambda}{4!}\right)^2 \times \underbrace{S}_{(4\times 3)^2\times 2} \frac{1}{2!} \int d^d z_1 \int d^d z_2 G_0(x_1, z_1) \times$$
$$\times G_0(x_2, z_1) G_0(x_3, z_2) G_0(x_4, z_2) \underbrace{\left(G_0(z_1, z_2)\right)^2}_{G_0(z_1,z_2)G_0(z_2,z_1)}$$

Note that:



is not topologically equivalent but gives the same result if $G_0(z_1, z_2) = G_0(z_2, z_1)$.

1.3.2 Feynman Rules In Momentum Space

If the Lagrangian $\mathcal{L}$ is translationally invariant we can take the following Fourier transforms:

$$\phi(x) = \int \frac{d^d k}{(2\pi)^d} e^{-\imath \mathbf{k}\cdot\mathbf{x}} \phi(\mathbf{k}) \tag{1.58}$$

$$J(x) = \int \frac{d^d k}{(2\pi)^d} e^{-\imath \mathbf{k}\cdot\mathbf{x}} J(\mathbf{k}) \tag{1.59}$$

Thus,

$$\mathcal{Z}_0(J) = \int \mathcal{D}\phi(\mathbf{k}) e^{-\frac{1}{2}\int \frac{d^d k}{(2\pi)^d}\left[\phi(\mathbf{k})\phi(-\mathbf{k})(k^2+m_0^2) - J(\mathbf{k})\phi(-\mathbf{k})\right]} \tag{1.60}$$

$$\int d^d x \mathcal{L}_{int} = \frac{\lambda}{4!}\int \frac{d^d k_1}{(2\pi)^d} \cdots \int \frac{d^d k_4}{(2\pi)^d} (2\pi)^d \delta^d(\sum_{i=1}^{4} \mathbf{k_i}) \phi(\mathbf{k}_1) \ldots \phi(\mathbf{k}_4) \tag{1.61}$$

and

$$G_N(x_1 \ldots x_N) = \int \frac{d^d p_1}{(2\pi)^d} \cdots \int \frac{d^d p_N}{(2\pi)^d} e^{-\imath \sum_{j=1}^{N} (\mathbf{p_j}\cdot\mathbf{x_j})} G_N(\mathbf{p}_1 \ldots \mathbf{p}_N) \tag{1.62}$$

$$G_N(p_1 \ldots p_N) = \langle \phi(p_1) \ldots \phi(p_N) \rangle = \frac{1}{\mathcal{Z}[J]} \frac{\delta^N \mathcal{Z}[J]}{\delta J(-p_1) \ldots \delta J(-p_N)}|_{J=0} \tag{1.63}$$

Also, the system's translation invariance leads to:

$$G_N(p_1 \ldots p_N) = (2\pi)^d \delta^d(\sum_{i=1}^{N} p_i) \bar{G}_N[\mathbf{p}] \tag{1.64}$$

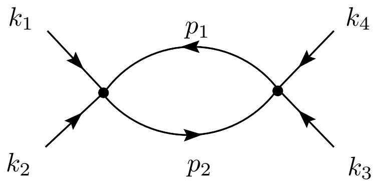
Also,

$$\mathcal{Z}[J] =$$

$$\mathcal{N} \exp\Big[-\frac{\lambda}{4!}\int \frac{d^d k_1}{(2\pi)^d} \cdots \int \frac{d^d k_4}{(2\pi)^d} (2\pi)^d \delta(\sum_{i=1}^{4} \mathbf{k}_i) \frac{\delta}{\delta J(-\mathbf{k}_1)} \cdots \frac{\delta}{\delta J(-\mathbf{k}_4)}\Big] \times \Big[e^{\frac{1}{2}\int \frac{d^d p}{(2\pi)^d} J(p) G_0(p) J(-p)} \Big] \tag{1.65}$$

where $G_0(p) = \frac{1}{p^2+m_0^2}$.

Example:



$$\equiv \left(-\frac{\lambda}{4!}\right)^2 \frac{1}{2!}(4\times 3)^2 \times 2\times 2G_0(k_1)G_0(k_2)G_0(k_3)G_0(k_4)$$
$$\times \int \frac{d^d p_1}{(2\pi)^d}\int \frac{d^d p_2}{(2\pi)^d}(2\pi)^d\delta^d(\mathbf{k}_1+\mathbf{k}_2+\mathbf{p}_1-\mathbf{p}_2)$$
$$\times(2\pi)^d\delta^d(\mathbf{k}_3+\mathbf{k}_4-\mathbf{p}_1+\mathbf{p}_2)G_0(p_1)G_0(p_2) \quad (1.66)$$
$$\equiv (2\pi)^d\delta^d(\mathbf{k}_1+\mathbf{k}_2+\mathbf{k}_3+\mathbf{k}_4)\left(-\frac{\lambda}{4!}\right)^2\frac{1}{2!}\times S\times 2\; G_0(k_1)\ldots G_0(k_4)$$
$$\times\int\frac{d^d p}{(2\pi)^d}G_0(p)G_0(|\mathbf{k}_1+\mathbf{k}_2-\mathbf{p}|) \quad (1.67)$$

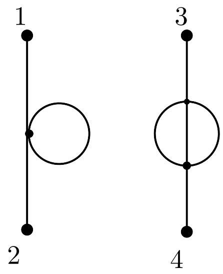
Thus the rules are very similar to those in position space.

1. *Construction of Graphs*: A general graph has N external points from each of which emanates a line(*a leg*) labelled $\mathbf{k}_i$ and n_r vertices of type r, represented by points from which r lines emanate labelled $\mathbf{q}_1\ldots\mathbf{q}_r$(one such set per vertex). All lines are connected pairwise and indices of a paired couple of lines are identical. No vacuum parts should be considered.

2. for every vertex of type r there's a factor $(2\pi)^d\left(-\frac{\lambda}{r!}\right)\delta(\sum_{i=1}^r \mathbf{q}_i)$, where all the $\mathbf{q}_i$ emanate from that vertex. For every line labelled $\mathbf{q}$ there is a factor $G_0(\mathbf{q})$. A multiplicity factor. Sum(integrate) over all internal momenta and indices.

1.4 Connected and Disconnected Green Functions

Suppose I want to compute $G_4(x_1,x_2,x_3,x_4)$. Obviously, there is a set of graphs where $G_4(x_1,x_2,x_3,x_4)\propto G_2(x_1,x_2)G_2(x_3,x_4)$+(other combinations).
e.g.,



note: this graph is linked (*i.e.,* it has no vacuum part), however it is disconnected

We saw that:

$$G_N(x_1, \ldots, x_N) = \frac{1}{\mathcal{Z}[\mathcal{J}]} \frac{\delta^{(N)} \mathcal{Z}[J]}{\delta J(x_1) \ldots \delta J(x_N)} \bigg|_{J=0} \tag{1.68}$$

Let's compute:

$$G_N^{(c)}(x_1, \ldots, x_N) = \frac{\delta^{(N)} \ln \mathcal{Z}[J]}{\delta J(x_1) \ldots \delta J(x_N)} \bigg|_{J=0} \tag{1.69}$$

Example:

$$G_2^{(c)}(x_1, x_2) = \frac{\delta^2 \ln \mathcal{Z}[J]}{\delta J(x_1) \delta J(x_2)} \bigg|_{J=0} \tag{1.70}$$

$$= \frac{\delta}{\delta J(x_1)} \frac{1}{\mathcal{Z}[J]} \frac{\delta \mathcal{Z}[J]}{\delta J(x_2)} \bigg|_{J=0} \tag{1.71}$$

$$= \frac{1}{\mathcal{Z}[J]} \frac{\delta^2 \mathcal{Z}[J]}{\delta J(x_1) \delta J(x_2)} \bigg|_{J=0} - \frac{1}{\mathcal{Z}^2[J]} \frac{\delta \mathcal{Z}[J]}{\delta J(x_1)} \bigg|_{J=0} \frac{\delta \mathcal{Z}[J]}{\delta J(x_2)} \bigg|_{J=0} \tag{1.72}$$

So,

$$G_2^{(c)}(x_1, x_2) = G_2(x_1, x_2) - G_1(x_1) G_1(x_2) \tag{1.73}$$

or

$$\langle\langle \phi(x_1)\phi(x_2) \rangle\rangle \equiv \langle \phi(x_1)\phi(x_2) \rangle_c \tag{1.74}$$

$$= \langle \phi(x_1)\phi(x_2) \rangle - \langle \phi(x_1) \rangle \langle \phi(x_2) \rangle \tag{1.75}$$

and the quantity

$$\langle \phi(x_1)\phi(x_2) \rangle_c = \langle [\phi(x_1) - \langle \phi(x_1) \rangle] [\phi(x_2) - \langle \phi(x_2) \rangle] \rangle \tag{1.76}$$

is called the *connected* Green function.
The generating functional of the *connected* Green function is the 'Free energy'(or

vacuum energy) $\mathcal{F}[J]$.

$$\mathcal{F}[J] = \ln \mathcal{Z}[J] \tag{1.77}$$

$$G_N^c(x_1,\ldots,x_N) = \frac{\delta^N \mathcal{F}[J]}{\delta J(x_1)\ldots\delta J(x_N)}|_{J=0} \tag{1.78}$$

are the '*cumulants*' or *connected* Green functions.
Remember that $J(x) \equiv H(x)$ is the magnetic field (e.g. in the Landau theory of magnetism). So,

$$\frac{\delta\mathcal{F}}{\delta J} \equiv \frac{d\mathcal{F}}{dH} = \langle\int d^dx\phi(x)\rangle = \int d^dx\langle\phi(x)\rangle = V\langle\phi\rangle \tag{1.79}$$

$$f = \frac{\mathcal{F}}{V} \Rightarrow \frac{df}{dH} = \langle\phi\rangle = M \to \text{Magnetization} \tag{1.80}$$

Also,

$$\frac{d^2f}{dH^2} = \frac{1}{V}\frac{d}{dH}\langle\int d^dx_1\phi(x_1)\rangle \tag{1.81}$$

$$=\frac{1}{V}\langle\int d^dx_1\int d^dx_2\phi(x_1)\phi(x_2)\rangle - \frac{1}{V}\langle\int d^dx_1\phi(x_1)\rangle\langle\int d^dx_2\phi(x_2)\rangle \tag{1.82}$$

$$\chi = \frac{d^2f}{dH^2} = \frac{1}{V}\int d^dx_1\int d^dx_2 G_2(x_1,x_2) - \frac{1}{V}V^2\langle\phi\rangle^2 \tag{1.83}$$

$$\chi = \frac{1}{V}V\int d^dy G_2(|\mathbf{y}|) - V\langle\phi\rangle^2 \tag{1.84}$$

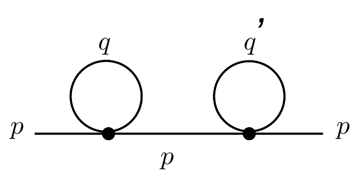
$$\chi = \int d^dy G_2^c(|\mathbf{y}|) \to \text{Susceptibility} \tag{1.85}$$

1.5 Vertex Functions

So far we have been able to reduce the number of diagrams to be considered by:

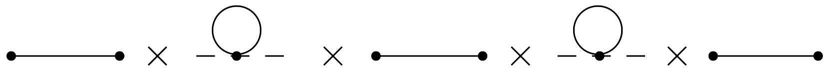
1. showing that vacuum parts do not contribute to $G_N(x_1,\ldots,x_N)$,
2. showing that disconnected parts need not be considered.

There is still another set of graphs that can be handled easily.
Consider the 2^{nd} order contribution to G_2:

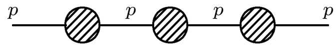


$$\equiv \left(-\frac{\lambda}{4!}\right)^2 \frac{1}{2!}(4\times 3)\cdot(4\times 3)G_0^3(p)\int d^dqG_0(q)\int d^dqG_0(q') \qquad (1.86)$$

Obviously this graph can be split in two by just cutting the middle line.



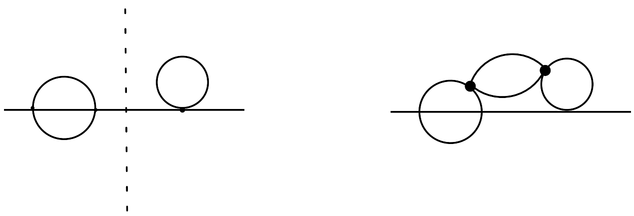
In general:



$$+\ldots \quad \Rightarrow \quad G_0^4(p)(\Sigma(p))^3+\ldots \qquad (1.87)$$

$$\begin{aligned}
G_2(p) &= G_0(p)+G_0(p)\Sigma(p)G_0(p)+G_0^3(p)\Sigma^2(p)+\ldots && (1.88)\\
G_2(p) &= G_0(p)\sum_{n=0}^{\infty}(\Sigma(p)G_0(p))^n = \frac{G_0(p)}{1-\Sigma(p)G_0(p)} && (1.89)\\
G_2^{-1}(p) &= G_0^{-1}(p)-\Sigma(p) && (1.90)\\
G_2(p) &= G_0(p)+G_0(p)\Sigma(p)G_2(p) \rightarrow \textit{Dyson's Equation} && (1.91)
\end{aligned}$$

where $\Sigma(p)$ represents the set of all possible connected, one-particle irreducible graphs with their external legs amputated.



One-Particle Reducible One-Particle Irreducible

The one-particle irreducible two-point function $\Sigma(p)$ is known as the mass operator or as the self-energy(or two-point vertex). Why?

$$G_0^{-1}(p) = p^2+m_0^2 \quad \overset{p\to 0}{\Rightarrow} \quad G_0^{-1}(0) = m_0^2 \qquad (1.92)$$

$$G_2^{-1}(p) = G_0^{-1}(p)-\Sigma(p) \quad \overset{p\to 0}{\Rightarrow} \quad G_2^{-1}(0) = m_0^2-\Sigma(0) = m^2 \qquad (1.93)$$

Thus, $\Sigma(0)$ renormalizes the mass. The result is in fact general:

$$[G_2^c(p)]^{-1} = G_0^{-1}(p)-\Sigma(p) \qquad (1.94)$$

1.5.1 General Vertex Functions

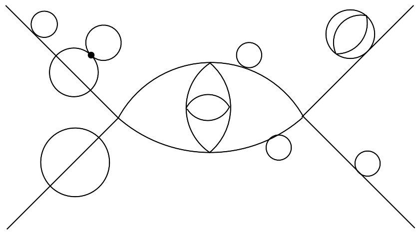


Figure 1.1: A 1-Particle Reducible vertex function

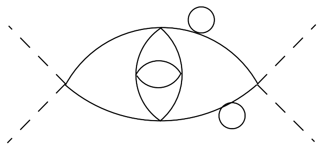


Figure 1.2: A 1-Particle Irreducible vertex function

We need a generating functional of 1-Particle Irreducible(1PI) vertex functions.
So far we have considered $\mathcal{F}[J]$ which is a function of the external sources. In many cases however, this is inconvenient since as $J \to 0$ we may still have $\langle\phi\rangle \neq 0$ (*symmetry breaking*). We would like to make a Legendre transformation from J to $\langle\phi\rangle$.
Let $\langle\phi(i)\rangle = \bar{\phi}(i) = \frac{\delta\mathcal{F}}{\delta J(i)}$ and thus, the Legendre transform $\Gamma\left[\bar{\phi}\right]$ can be defined as follows:

$$\Gamma[\bar{\phi}] = \sum_i \bar{\phi}(i)J(i) - \mathcal{F}[J] \tag{1.95}$$

$$\begin{aligned}\Rightarrow \frac{\delta\Gamma[\bar{\phi}]}{\delta\bar{\phi}(i)} &= -\sum_j \frac{\delta\mathcal{F}}{\delta J(j)}\frac{\delta J(j)}{\delta\bar{\phi}(i)} + \sum_j \bar{\phi}(j)\frac{\delta J(j)}{\delta\bar{\phi}(i)} + \sum_j J(j)\delta(i,j) &(1.96)\\ &= -\sum_j \bar{\phi}(j)\frac{\delta J(j)}{\delta\bar{\phi}(i)} + \sum_j \bar{\phi}(j)\frac{\delta J(j)}{\delta\bar{\phi}(i)} + J(i) &(1.97)\end{aligned}$$

Thus,

$$\frac{\delta\Gamma[\bar{\phi}]}{\delta\bar{\phi}(i)} = J(i) \tag{1.98}$$

However, if $J \to 0$ still $\frac{\delta \mathcal{F}}{\delta J(i)}|_{J=0} = \bar{\phi}(i)$, then the symmetry is broken if $\bar{\phi}(i) \neq 0$. Thus $\bar{\phi}(i)$ satisfies $\frac{\delta \Gamma[\bar{\phi}]}{\delta \bar{\phi}(i)} = 0$ and it minimizes the potential Γ. $\bar{\phi}(i)$ is known as the *classical field*. Let's differentiate the classical field by $\bar{\phi}(j)$:

$$\delta(i,j) = \frac{\delta^2 \mathcal{F}}{\delta J(i) \delta \bar{\phi}(j)} = \sum_k \frac{\delta^2 \mathcal{F}}{\delta J(i) \delta J(k)} \frac{\delta J(k)}{\delta \bar{\phi}(j)} \tag{1.99}$$

$$\delta(i,j) = \sum_k \frac{\delta^2 \mathcal{F}}{\delta J(i) \delta J(k)} \frac{\delta^2 \Gamma}{\delta \bar{\phi}(k) \delta \bar{\phi}(j)} \tag{1.100}$$

As $J \to 0$

$$\frac{\delta^2 \mathcal{F}}{\delta J(i) \delta J(k)} \to G_2^c(i,k) \tag{1.101}$$

So, $\Gamma^{(2)}(i,j) = \frac{\delta^2 \Gamma}{\delta \bar{\phi}(i) \bar{\phi}(j)}|_{J=0}$ is the inverse matrix of $G_2^c(i,j)$. Therefore, in momentum space:

$$\Gamma^{(2)}(p) = [G_2^c(p)]^{-1} = p^2 + m_0^2 - \Sigma(p) \tag{1.102}$$

Thus, $\Gamma^{(2)}(p)$ is a sum of 1PI graphs.
Now, if we differentiate (1.100) by $J(l)$ we will have:

$$\begin{aligned} \frac{\delta}{\delta J(l)} \delta(i,j) = 0 = & \sum_k \frac{\delta^3 \mathcal{F}}{\delta J(i) \delta J(k) \delta J(l)} \frac{\delta J(k)}{\delta \bar{\phi}(j)} \\ & + \sum_k \frac{\delta^2 \mathcal{F}}{\delta J(i) \delta J(k)} \frac{\delta^2 J(k)}{\delta J(l) \delta \bar{\phi}(j)} \end{aligned} \tag{1.103}$$

But,

$$\frac{\delta^2 J(k)}{\delta J(l) \delta \bar{\phi}(j)} = \sum_m \frac{\delta^3 \Gamma}{\delta \bar{\phi}(m) \delta \bar{\phi}(k) \delta \bar{\phi}(j)} \frac{\delta \bar{\phi}(m)}{\delta J(l)} \tag{1.104}$$

$$= \sum_m \frac{\delta^3 \Gamma}{\delta \bar{\phi}(m) \delta \bar{\phi}(k) \delta \bar{\phi}(j)} \frac{\delta^2 \mathcal{F}}{\delta J(l) \delta J(m)} \tag{1.105}$$

So,

$$0 = \sum_k G_3^c(i,k,l) \Gamma^{(2)}(k,j) + \sum_{k,m} G_2^c(i,k) G_2^c(l,m) \Gamma^{(3)}(m,k,j) \tag{1.106}$$

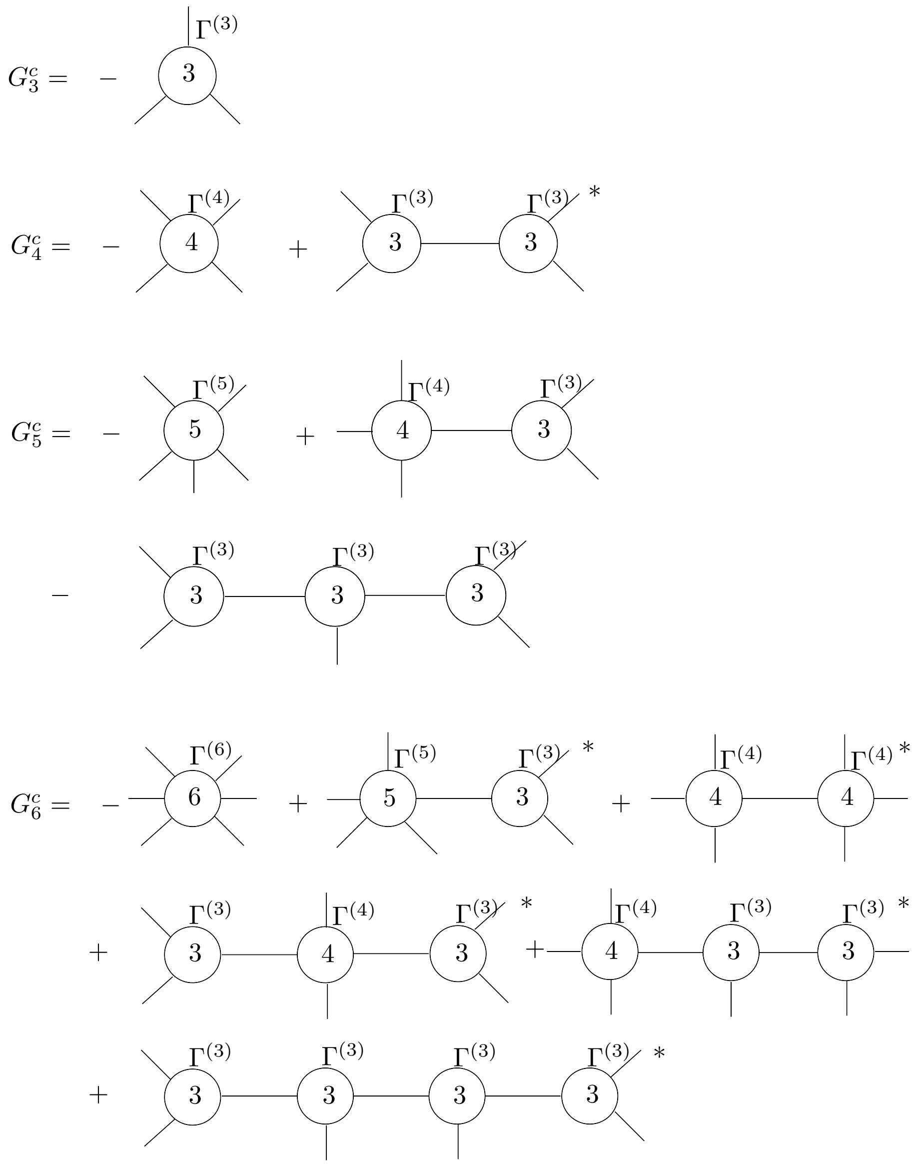
where $\Gamma^{(2)} = [G_2]^{-1}$.

$$G_3^c(i_1,i_2,i_3) = -G_2^c(i_1,j_1) G_2^c(i_2,j_2) G_2^c(i_3,j_3) \Gamma^{(3)}(j_1,j_2,j_3) \tag{1.107}$$

Note that:

$$G_2^c(i_1,i_2) = G_2^c(i_1,j_1) G_2^c(i_2,j_2) \Gamma^{(2)}(j_1,j_2) \tag{1.108}$$

since $G_2^c = [\Gamma^{(2)}]^{-1}$.
Thus $\Gamma^{(3)}$ is the 1PI 3-point vertex .
Pictorially,



etc.

Note: The symbol (*) means that the respective diagram is One-Particle Reducible by a body cut. Also, in each diagram, summation over all possible equivalent combinations is implied.

Clearly, a graph may be reducible either by a cut of only an external line or via a body cut.
From the definition of $\Gamma^{(N)}$ we have:

$$\Gamma^{(N)}(1,\ldots,N) = \frac{\delta^N \Gamma(\bar{\phi})}{\delta\bar{\phi}(1)\ldots\delta\bar{\phi}(N)}|_{J=0} \tag{1.109}$$

$$G_N^c(1,\ldots,N) \overset{N\geq 2}{=} -G_2^c(1,1')\ldots G_2^c(N,N')\Gamma^{(N)}(1',\ldots,N') + Q^{(N)}(1,\ldots,N) \tag{1.110}$$

where the 1^{st} terms are 1PR only via cuts of the external legs and the 2^{nd} by body cuts(for the r-point function in a ϕ^r theory this term does not exist).
Momentum Space:

$$G_2^c(k_1,k_2) = \delta^d(\mathbf{k}_1+\mathbf{k}_2)G_2^c(k_1)(2\pi)^d \tag{1.111}$$

and by using (1.102):

$$\Gamma^{(2)}(k_1,k_2) = (2\pi)^d\delta^d(\mathbf{k}_1+\mathbf{k}_2)\Gamma^{(2)}(k_1) \tag{1.112}$$

and

$$G_N^c(k_1,\ldots,k_N) \overset{N\geq 2}{=} -G_2^c(k_1)\ldots G_2^c(k_N)\Gamma^{(N)}(k_1,\ldots,k_N) + Q^{(N)}(k_1,\ldots,k_N) \tag{1.113}$$

1.6 The Effective Potential

Let $\upsilon = \bar{\phi} = \langle\phi\rangle$. Then, with the above definition for $\Gamma^{(N)}$ we may write:

$$\Gamma\left[\bar{\phi}\right] = \\ = \sum_{N=1}^{\infty}\frac{1}{N!}\int d^dx_1\ldots\int d^dx_N\Gamma^{(N)}(x_1,\ldots,x_N|\upsilon)[\bar{\phi}(x_1)-\upsilon]\ldots[\bar{\phi}(x_N)-\upsilon] \tag{1.114}$$

If $J\to 0$ then the sum starts at N=2, $\upsilon = \lim_{J\to 0}\bar{\phi}$ and also it follows that υ is a local minimum of Γ, from the following facts:

$$\frac{\delta\Gamma}{\delta\bar{\phi}} = J \overset{J\to 0}{=} 0 \tag{1.115}$$

$$\Gamma^{(2)} \geq 0 \tag{1.116}$$

But for a general function:

$$f(z) = \sum_{n=0}^{\infty}\frac{1}{n!}f^{(n)}(a)(z-a)^n \quad \text{around z=a} \tag{1.117}$$

$$f(z) = \sum_{n=0}^{\infty}\frac{1}{n!}f^{(n)}(0)z^n \quad \text{around z=0} \tag{1.118}$$

So, in the symmetric theory:

$$\Gamma\left[\bar{\phi}\right] = \sum_{N=1}^{\infty} \frac{1}{N!} \int d^d x_1 \ldots \int d^d x_N \Gamma^{(N)}(x_1, \ldots, x_N) \bar{\phi}(x_1) \ldots \bar{\phi}(x_N) \quad (1.119)$$

The classical field $\bar{\phi}$ is defined by $\frac{\delta\Gamma}{\delta\bar{\phi}} = 0$ and if $\bar{\phi} \neq 0$ then the symmetry is spontaneously broken. Moreover, for $\langle\phi\rangle = \bar{\phi} = const.$:

$$\Gamma\left[\bar{\phi}\right] = \sum_{N=2}^{\infty} \frac{1}{N!} \left[\int d^d x_1 \ldots \int d^d x_N \Gamma^{(N)}(x_1, \ldots, x_N)\right] \bar{\phi}^N \quad (1.120)$$

and

$$\Gamma^{(N)}(x_1, \ldots, x_N) = \int \frac{d^d k_1}{(2\pi)^d} \cdots \int \frac{d^d k_N}{(2\pi)^d} \Gamma^{(N)}(k_1, \ldots, k_N) e^{-\imath \mathbf{k}_j \cdot \mathbf{x}_j} \quad (1.121)$$

But $\Gamma^{(N)}(k_1, \ldots, k_N) = (2\pi)^d \delta^d(\sum_j \mathbf{k}_j) \tilde{\Gamma}^{(N)}(k_1, \ldots, k_N)$, so we finally have:

$$\Gamma(\bar{\phi}) = \sum_{N=2}^{\infty} \frac{1}{N!} \tilde{\Gamma}^{(N)}(0, \ldots, 0) \bar{\phi}^N \underbrace{(2\pi)^d \delta^d(0)}_{\equiv V} \quad (1.122)$$

$$\Gamma(\bar{\phi}) = V U(\bar{\phi}) \quad (1.123)$$

$$U(\bar{\phi}) = \sum_{N=2}^{\infty} \frac{1}{N!} \tilde{\Gamma}^{(N)}(0, \ldots, 0) \bar{\phi}^N \rightarrow \text{Effective Potential} \quad (1.124)$$

Note that the $\tilde{\Gamma}^{(N)}(0, \ldots, 0)$'s are computed in the *symmetric theory.* In this framework, there will be *symmetry breaking* if U has a minimum at $\bar{\phi} \neq 0$. If we identify $J(x) \equiv H$ with the external physical field, then it follows from (1.98) that:

$$\frac{dU}{d\bar{\phi}} = H \quad (1.125)$$

From the above relation(1.125), the equation of state follows:

$$H = \sum_{N=2}^{\infty} \frac{1}{N!} N \tilde{\Gamma}^{(N)}(0, \ldots, 0) \bar{\phi}^{N-1} \quad (1.126)$$

$$H = \sum_{N=1}^{\infty} \frac{1}{N!} \tilde{\Gamma}^{(N+1)}(0, \ldots, 0) \bar{\phi}^N \quad (1.127)$$

Thus we first must compute the effective potential and from it the vacuum (ground state). Next one computes the full vertex functions, either in the symmetric or broken symmetry state, by identifying in $\Gamma[\bar{\phi}]$ the coefficients of the products $\prod_i (\bar{\phi}(x_i) - \upsilon)$, where υ is the classical field which minimizes $U(\bar{\phi})$, i.e.

$$\Gamma^{(N)}(1, \ldots, N|\upsilon) = \frac{\delta^N \Gamma[\bar{\phi}]}{\delta\bar{\phi}(1) \ldots \delta\bar{\phi}(N)}|_{\bar{\phi}=\upsilon} \quad (1.128)$$

1.7 Ward Identities

I want to discuss now the consequences of the existence of a continuous symmetry (e.g. O(2)).
Let $\vec{\phi}(x) = \begin{bmatrix} \phi_\pi(x) \\ \phi_\sigma(x) \end{bmatrix}$ and also:

$$\mathcal{L}(\vec{\phi}) = \frac{1}{2}\left[(\nabla\vec{\phi})^2 + m_0^2\vec{\phi}^2\right] + \frac{\lambda}{4!}(\vec{\phi}^2)^2 \tag{1.129}$$

and the Lagrangian $\mathcal{L}$ has an O(2) symmetry. This means that $\mathcal{L}$ remains invariant under the following transformation:

$$\vec{\phi}' = \left(\begin{array}{cc} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{array}\right)\vec{\phi} \equiv \vec{\phi} \tag{1.130}$$

For θ infinitesimal :

$$T = I + \epsilon\left(\begin{array}{cc} 0 & -1 \\ 1 & 0 \end{array}\right) \tag{1.131}$$

The generating functional is invariant if the sources are transformed accordingly : $\vec{J}' = T\vec{J}$ since T is orthogonal. So $\vec{J}\cdot\vec{\phi}$ is invariant and also the measure is invariant. We have:

$$\vec{J}' = \vec{J} + \epsilon\left(\begin{array}{cc} 0 & -1 \\ 1 & 0 \end{array}\right)\left(\begin{array}{c} J_\pi \\ J_\sigma \end{array}\right) \tag{1.132}$$

$$\left\{\begin{array}{l} J'_\pi = J_\pi - \epsilon J_\sigma \\ J'_\sigma = J_\sigma + \epsilon J_\pi \end{array}\right. \Rightarrow \left\{\begin{array}{l} \delta J_\pi = -\epsilon J_\sigma \\ \delta J_\sigma = +\epsilon J_\pi \end{array}\right. \tag{1.133}$$

Since $\mathcal{F}[\vec{J}]$ is invariant, we have:

$$\delta\mathcal{F} = \int d^dx\left[\frac{\delta\mathcal{F}[\vec{J}]}{\delta J_\sigma(x)}\delta J_\sigma(x) + \frac{\delta\mathcal{F}[\vec{J}]}{\delta J_\pi(x)}\delta J_\pi(x)\right] = 0 \tag{1.134}$$

$$\delta\mathcal{F} = \int d^dx\,\epsilon\left[\frac{\delta\mathcal{F}[\vec{J}]}{\delta J_\sigma(x)}J_\pi(x) - \frac{\delta\mathcal{F}[\vec{J}]}{\delta J_\pi(x)}J_\sigma(x)\right] = 0 \tag{1.135}$$

or

$$\int d^dx\left[\bar{\phi}_\sigma(x)J_\pi(x) - \bar{\phi}_\pi(x)J_\sigma(x)\right] = 0 \tag{1.136}$$

$$\int d^dx\left[\bar{\phi}_\sigma(x)\frac{\delta\Gamma[\bar{\phi}]}{\delta\bar{\phi}_\pi(x)} - \bar{\phi}_\pi(x)\frac{\delta\Gamma[\bar{\phi}]}{\delta\bar{\phi}_\sigma(x)}\right] = 0 \tag{1.137}$$

The above equation is called *Ward Identity* and says that $\Gamma[\bar{\phi}]$ is invariant under $\vec{\phi} \to T\vec{\phi}$. This identity is always valid(i.e. to all orders in perturbation theory).

$$0 = \frac{\delta}{\delta\bar{\phi}_\pi(y)} \left[\int d^dx \left\{ \frac{\delta\Gamma}{\delta\bar{\phi}_\pi}\bar{\phi}_\sigma - \frac{\delta\Gamma}{\delta\bar{\phi}_\sigma}\bar{\phi}_\pi \right\} \right] \tag{1.138}$$

$$0 = \int d^dx \Big\{ \frac{\delta^2\Gamma}{\delta\bar{\phi}_\pi(y)\delta\bar{\phi}_\pi(x)}\bar{\phi}_\sigma(x) - \frac{\delta^2\Gamma}{\delta\bar{\phi}_\sigma(y)\delta\bar{\phi}_\pi(x)}\bar{\phi}_\pi(x) - \frac{\delta\Gamma}{\delta\bar{\phi}_\sigma(x)}\delta^d(x-y) \Big\} \tag{1.139}$$

$$\frac{\delta\Gamma}{\delta\bar{\phi}_\sigma(y)} = \int d^dx \left[\frac{\delta^2\Gamma}{\delta\bar{\phi}_\pi(x)\delta\bar{\phi}_\pi(y)}\bar{\phi}_\sigma(x) - \frac{\delta^2\Gamma}{\delta\bar{\phi}_\sigma(x)\delta\bar{\phi}_\pi(y)}\bar{\phi}_\pi(x) \right] \tag{1.140}$$

If the symmetry is broken, say $\bar{\phi} = \begin{bmatrix} 0 \\ u \end{bmatrix}$, the above equation yields:

$$u \int d^dx \frac{\delta^2\Gamma}{\delta\bar{\phi}_\pi(x)\delta\bar{\phi}_\pi(y)} = J_\sigma \tag{1.141}$$

then if $\begin{pmatrix} J_\pi \\ J_\sigma \end{pmatrix} \to 0$ and $\frac{\delta^2\Gamma}{\delta\bar{\phi}_\pi(x)\delta\bar{\phi}_\pi(y)} \to \Gamma^{(2)}_{\pi\pi}(x-y)$:

$$u \int d^dx \Gamma^{(2)}_{\pi\pi}(x-y) \overset{J\to 0}{=} 0 \tag{1.142}$$

or

$$u \int d^dx \Gamma^{(2)}_{\pi\pi}(x-y) \overset{J=\left[\begin{smallmatrix}0\\H\end{smallmatrix}\right]}{=} H \tag{1.143}$$

$$\lim_{\mathbf{p}\to 0} u\bar{\Gamma}^{(2)}_{\pi\pi}(\mathbf{p}) = H \tag{1.144}$$

$$\Gamma^{(2)}_{\pi\pi}(\mathbf{p}) \overset{\mathbf{p}\to 0}{=} \frac{H}{u} \tag{1.145}$$

so, if $u \neq 0$ then $\Gamma^{(2)}_{\pi\pi} \to 0$ as $H \to 0$. Therefore, $G^c_{2,\pi\pi}(\mathbf{p})$ has a pole at zero momentum in the *spontaneously* broken phase. This is the famous *Goldstone boson.*

Thus we discover that there is an alternative: either

1. the theory is symmetric, i.e. $u = 0$ or

2. the symmetry is spontaneously broken, $u \neq 0$ with $J \to 0$, and there are massless excitations (*Nambu-Goldstone bosons*).

In the O(N) case we have $\frac{N(N-1)}{2}$ generators:

$$(L_{ij})_{kl} = -i\left[\delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk}\right] \tag{1.146}$$

If the symmetry is spontaneously broken, we have:

$$\bar{\phi} = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ u \end{bmatrix} \quad \text{and in general} \quad \bar{\phi} = \begin{bmatrix} \pi_1 \\ \vdots \\ \pi_{N-1} \\ \sigma \end{bmatrix} = \begin{bmatrix} \vec{\pi} \\ \sigma \end{bmatrix} \tag{1.147}$$

Obviously, the N-1 components have a symmetry O(N-1). Thus, the symmetry which is actually broken is the $\frac{O(N)}{O(N-1)} \equiv S_N$ (N-dimensional sphere) rather than O(N).

$$\bar{\phi}'_a = \left[e^{i\vec{\lambda}\cdot\vec{L}}\right]_{ab} \bar{\phi}_b \qquad (\lambda_{ij} = -\lambda_{ji}) \tag{1.148}$$

The broken generators are L_{in} and L_{ij} (with i,j≠n) are the generators of the unbroken O(N-1) symmetry. Let's use the infinitesimal transformation with the L_{in}'s:

$$\bar{\phi}'_a = \left[e^{i\lambda_{in}L_{in}}\right]_{ab} \bar{\phi}_b \tag{1.149}$$

$$\delta\bar{\phi}_a = i\lambda_{in}[L_{in}]_{ab}\bar{\phi}_b \tag{1.150}$$

$$\delta\bar{\phi}_a = \lambda_{in}[\delta_{ia}\delta_{nb} - \delta_{ib}\delta_{na}]\bar{\phi}_b \tag{1.151}$$

$$\delta\bar{\phi}_a = \lambda_{an}\bar{\phi}_n - \lambda_{bn}\delta_{na}\bar{\phi}_b \tag{1.152}$$

or

$$\left\{ \begin{array}{lcl} \delta\sigma & = & -\lambda_{bn}\pi_b \\ \delta\pi_a & = & \lambda_{an}\sigma \end{array} \right. \Rightarrow \left\{ \begin{array}{lcl} \delta\sigma & = & \lambda_{nb}\pi_b \\ \delta\pi_a & = & -\lambda_{na}\sigma \end{array} \right. \tag{1.153}$$

Likewise,

$$\begin{array}{lcl} \delta J_\sigma & = & -\lambda_{nb}J_{\pi_b} \\ \delta J_{\pi_a} & = & +\lambda_{na}J_\sigma \end{array} \tag{1.154}$$

Thus,

$$\delta\mathcal{F} = 0 = \int d^dx \left[\frac{\delta\mathcal{F}}{\delta J_\sigma(x)}\delta J_\sigma + \frac{\delta\mathcal{F}}{\delta J_{\pi_a(x)}}\delta J_{\pi_a}(x)\right] \tag{1.155}$$

$$0 = \int d^dx \left[-\lambda_{nb}\sigma(x)J_{\pi_b}(x) + \lambda_{na}J_\sigma(x)\pi_a(x)\right] \tag{1.156}$$

$$0 = \int d^dx\lambda_{na}\left[J_\sigma(x)\pi_a(x) - \sigma(x)J_{\pi_a}(x)\right] \tag{1.157}$$

$$0 = \int d^dx\lambda_{na}\left[\frac{\delta\Gamma}{\delta\sigma(x)}\pi_a(x) - \frac{\delta\Gamma}{\delta\pi_a(x)}\sigma(x)\right] \tag{1.158}$$

since λ_{na} are arbitrary, we have for each a:

$$0 = \int d^dx \left[\frac{\delta\Gamma}{\delta\sigma(x)}\pi_a(x) - \frac{\delta\Gamma}{\delta\pi_a(x)}\sigma(x)\right] \rightarrow \text{Ward Identity} \tag{1.159}$$

and

$$0 = \frac{\delta}{\delta\pi_b(y)} \int d^d x \left[\frac{\delta\Gamma}{\delta\sigma(x)} \pi_a(x) - \frac{\delta\Gamma}{\delta\pi_a(x)} \sigma(x) \right] \quad (1.160)$$

$$0 = \int d^d x \left[\frac{\delta^2\Gamma}{\delta\sigma(x)\delta\pi_b(y)} \pi_a(x) + \frac{\delta\Gamma}{\delta\sigma(x)} \delta_{ab}\delta(x-y) - \frac{\delta^2\Gamma}{\delta\pi_a(x)\delta\pi_b(y)} \sigma(x) \right] \quad (1.161)$$

If $\bar{\phi} = \begin{bmatrix} 0 \\ u \end{bmatrix}$, then

$$\delta_{ab} \frac{\delta\Gamma}{\delta\sigma(y)} = u \int d^d x \frac{\delta^2\Gamma}{\delta\pi_a(x)\delta\pi_b(y)} \quad (1.162)$$

and

$$\delta_{ab} J_\sigma = u \lim_{p\to 0} \Gamma^{(2)}_{\pi_a\pi_b}(p) \quad (1.163)$$

From the above equation, the following conclusions can be made:

1. $\Gamma_{\pi_a\pi_b}$ must be diagonal $\Gamma^{(0)}_{\pi_a\pi_b} = \delta_{ab}\Gamma_{\pi\pi}(0)$ and all masses are equal, $m^2_{\pi_a} = m^2_{\pi_b}$(degenerate multiplet)
2. $J_\sigma \to 0$ leads to: $u \lim_{p\to 0} \Gamma_{\pi_a\pi_a}(p) = 0$ and thus,
   (a) If $u \neq 0$, all π excitations are massless, leading to N-1 Goldstone bosons.
   (b) If $u = 0$, then the theory is symmetric.

These results are valid order by order in perturbation theory. We can get, in fact, an infinite set of identities. For example, in an O(2) theory, by differentiating (1.139) with respect to the σ-field:

$$0 = \frac{\delta}{\delta\sigma(z)} \int d^d x \left[\sigma(x)\Gamma^{(2)}_{\pi\pi}(x,y) - \delta(x-y)\Gamma_\sigma(x) - \pi(x)\Gamma_{\sigma\pi}(x,y) \right] \quad (1.164)$$

$$0 = \int d^d x \Big[\delta(x-z)\Gamma^{(2)}_{\pi\pi}(x,y) + \sigma(x)\Gamma^{(3)}_{\pi\pi\sigma}(x,y,z) - \delta(x-y)\Gamma^{(2)}_{\sigma\sigma}(x,z) - \pi(x)\Gamma_{\sigma\pi\sigma}(x,y,z) \Big] \quad (1.165)$$

$$0 = \Gamma^{(2)}_{\pi\pi}(z,y) - \Gamma^{(2)}_{\sigma\sigma}(y,z) + u \int d^d x \Gamma^{(3)}_{\pi\pi\sigma}(x,y,z) \quad (1.166)$$

So, finally,

$$\Gamma^{(2)}_{\sigma\sigma}(p) - \Gamma^{(2)}_{\pi\pi}(p) = u \int d^d x \Gamma^{(3)}_{\pi\pi\sigma}(0,p,-p) \quad (1.167)$$

Likewise, if $u = 0$ then $\Gamma_{\sigma\sigma}(p) = \Gamma_{\pi\pi}(p)$(symmetric) and if $u \neq 0$ then $\Gamma^{(2)}_{\sigma\sigma}(0) = u\Gamma^{(3)}_{\pi\pi\sigma}(0,0,0)$.

Also, by differentiating (1.165) with respect to the σ-field two more times and using the identity (1.167), we can derive one more identity:

$$\Gamma^{(4)}_{\pi\pi\sigma\sigma}(z,y,t,w)+\Gamma^{(4)}_{\pi\pi\sigma\sigma}(w,y,z,t)+\Gamma^{(4)}_{\pi\pi\sigma\sigma}(t,y,z,w)=\Gamma^{(4)}_{\sigma\sigma\sigma\sigma}(y,z,t,w) \quad (1.168)$$

The Fourier transform of the above identity at a symmetric point satisfies: (e.g. for $p \to 0$)

$$3\Gamma^{(4)}_{\pi\pi\sigma\sigma}|_{S.P.}=\Gamma^{(4)}_{\sigma\sigma\sigma\sigma}|_{S.P.} \quad (1.169)$$

Thus, rotational invariance is guaranteed.

1.8 The Loop expansion

The pertrbative expansion developed above seems to be limited to situations in which:

1. λ is small and
2. the vacuum is symmetric ($\langle\phi\rangle = 0$)

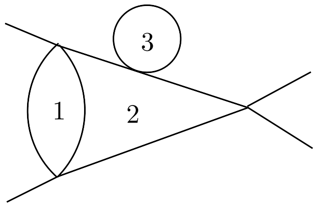
There are, however, other expansion schemes. One such scheme is the WKB method in Quantum Mechanics, even though it is difficult to generalize to a field theory as it stands.
There is another way to proceed:

$$\begin{aligned}
\mathcal{Z}[J] &= \int \mathcal{D}\phi e^{\frac{\mathcal{S}[\phi]}{\alpha}} && (1.170)\\
&= \int \mathcal{D}\phi e^{-\frac{1}{\alpha}\int d^dx \mathcal{L}_{int}[\frac{\delta}{\delta J}]} \times \int \mathcal{D}\phi e^{-\frac{1}{\alpha}\int d^dx \mathcal{L}_0[\phi]+\int d^dx J\cdot\phi} && (1.171)\\
&= \int \mathcal{D}\phi e^{-\frac{1}{\alpha}\int d^dx \mathcal{L}_{int}[\frac{\delta}{\delta J}]} \times \int \mathcal{D}\phi e^{-\frac{1}{2\alpha}\int d^dx \phi G^{-1}\phi+\int d^dx J\cdot\phi} && (1.172)\\
&= \int \mathcal{D}\phi e^{-\frac{1}{\alpha}\int d^dy \mathcal{L}_{int}[\frac{\delta}{\delta J(y)}]} \times e^{\frac{\alpha}{2}\int d^dx \int d^dx' J(x)G_0(x,x')J(x')} && (1.173)
\end{aligned}$$

Thus the diagrammatic rules are the same as before, with the additions:

1. every vertex acquires a weight $\frac{1}{\alpha}$
2. every propagator acquires a factor α

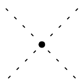
Thus, a graph with N external points and I internal lines will have a weight (to order n) α^{I-n}.
How many momentum integrations? There are n $\delta-$functions but one of them expresses the overall momentum conservation, so the number L of independent momentum integrations in each diagram is $L = I-(n-1)$ so the weight of a graph becomes $\alpha^{I-n} = \alpha^{L-1}$. Thus we really are expanding in powers of the number of intependent integrals or *loops*:



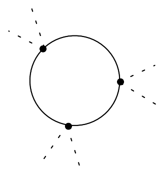
$\to a^{3-1} = a^2$

$N = 4$
$n = 4$
$I = 6$
$L = 3 = 6 - (4-1) = 3$

In this case, though, it coincides with an expansion in powes of λ. If we had several coupling constants, the situation would be totally different.

1. *The tree approximation (no loops):*
   Let's compute Γ to this order:
   (a) $\Gamma^{(2)}(k_1, k_2) = (2\pi)^d \delta^d(k_1+k_2)(k_1^2+m_0^2)$ because $\Gamma^{(2)} = G_0^{-1} - \Sigma$ and $\Sigma = 0$ at the tree level, since all 1PI graphs contain at least one loop.
   (b) $\Gamma^{(3)} = 0$
   (c) $\Gamma^{(4)}(k_1, \ldots, k_4) =$  $= (2\pi)^d \delta^d(k_1 + \ldots + k_4)\lambda$

   All higher vertices contain, at least, one loop, e.g. (in ϕ^4 theory):

   $\Gamma^{(6)} =$  $+ \ldots$

   So, at the tree level:

$$\Gamma\{\bar\phi\} = \sum_{N=1}^{\infty} \frac{1}{N!} \int \frac{d^d q_1}{(2\pi)^d} \cdots \int \frac{d^d q_N}{(2\pi)^d} \Gamma^{(N)}(q_1, \ldots, q_N)\bar\phi(-q_1)\ldots\bar\phi(-q_N) \quad (1.174)$$

$$\equiv \frac{1}{2!} \int \frac{d^d q_1}{(2\pi)^d} \int \frac{d^d q_2}{(2\pi)^d} (2\pi)^d \delta^d(q_1+q_2)(k_1^2+m_0^2)\bar\phi(-q_1)\bar\phi(-q_2) \quad (1.175)$$

$$+\frac{1}{4!}\lambda \int \frac{d^d q_1}{(2\pi)^d} \cdots \int \frac{d^d q_4}{(2\pi)^d} (2\pi)^d \delta^d(q_1 + \ldots + q_4)$$
$$\times (k_1^2+m_0^2)\bar\phi(-q_1)\ldots\bar\phi(-q_4) + O(\lambda^2) \quad (1.176)$$

   Thus,

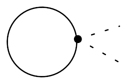
$$\Gamma_0(\bar\phi) = \int d^d x \left[\frac{1}{2}(\nabla\bar\phi)^2 + \frac{m_0^2}{2}\bar\phi^2 + \frac{\lambda}{4!}(\bar\phi^2)^2\right] \quad (1.177)$$

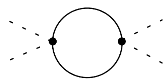
   which is just the Landau theory.
   If $m_0^2 > 0$, then $\Phi = 0$ but for $m_0^2 < 0$, $\Phi = \pm\sqrt{-\frac{6m_0^2}{\lambda}}$.

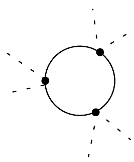
2. *One-Loop Corrections:*
L=1 leads to I=n (*i.e.,* (number of internal lines) = (order of Pert.Exp)). In a ϕ^4 theory a graph with N external points, I internal lines and order n satisfies:

$$4n = N + 2I \tag{1.178}$$

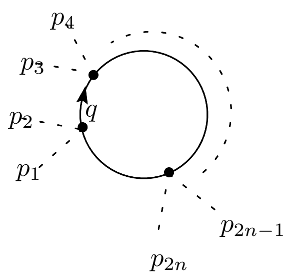
i.e., the lines emerging from all vertices must either be tied up together or attached to an external point)
So, L=1 means I=n and therefore, $n = \frac{N}{2}$. Thus, the one-loop corrections are: and a contribution to $\Gamma^{(N)}$ has the form:

$\Gamma^{(2)} =$ 

$\Gamma^{(4)} =$ 

$\Gamma^{(6)} =$ 

(N=2n)



$$\bar{\Gamma}_1^N(0,\ldots,0) = -(-\frac{\lambda}{4!})^n \frac{1}{n!} S_n \int \frac{d^dq}{(2\pi)^d} \left[\frac{1}{q^2+m_0^2}\right]^n \times (N-1)! \tag{1.179}$$

where $S_n = (4 \times 3)^n \times n!$, n! being the number of ways of reordering the vertices and (N-1)! being the number of ways of attaching p's to the external vertices.

Now, we may compute the corrections to the effective potential:

$$\Gamma_1[\Phi] = \sum_{N=1}^{\infty} \frac{1}{N!} \Phi^N \bar{\Gamma}_1^{(N)}(0,\ldots,0)(2\pi)^d \delta^d(0) \tag{1.180}$$

$$U_1[\Phi] =$$

$$= -\sum_{n=1}^{\infty} \frac{1}{(2n)!} \Phi^{2n} (-\frac{\lambda}{4!})^n \frac{(4\times 3)^n}{n!} n!(2n-1)! \int \frac{d^dq}{(2\pi)^d} \left[\frac{1}{q^2+m_0^2}\right]^n \tag{1.181}$$

$$U_1[\Phi] = -\sum_{n=1}^{\infty} \frac{1}{2n} \int \frac{d^dq}{(2\pi)^d} \left[-\frac{\lambda\Phi^2}{q^2+m_0^2}\right]^n \tag{1.182}$$

and by the definition of the logarithm $\ln(1+x) = -\sum_{N=1}^{\infty} \frac{(-x)^n}{n}$ we take:

$$U_1[\Phi] = \frac{1}{2} \int \frac{d^dq}{(2\pi)^d} \ln\left[1 + \frac{\lambda\Phi^2}{2(q^2+m_0^2)}\right] \tag{1.183}$$

So, the effective potential to the order of one loop is:

$$U[\Phi] = \frac{m_0^2}{2}\Phi^2 + \frac{\lambda}{4!}\Phi^4 + \frac{1}{2}\int \frac{d^dq}{(2\pi)^d} \ln\left[q^2 + m_0^2 + \frac{\lambda\Phi^2}{2}\right] \underbrace{- \frac{1}{2}\int \frac{d^dq}{(2\pi)^d} \ln\left[q^2+m_0^2\right]}_{\text{const.energy shift}} \tag{1.184}$$