

Physics 583, Academic Year 2008/2009
Spring Semester 2009
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Problem Set No. 1
Due Date: February 27, 2009

1 Vertex Functions, Effective Potential and Ward Identities

In this problem you will study the effects of interactions on the physical properties of a system of interacting relativistic fermions in $1 + 1$ dimensions. This model is a reasonable description of the physics in a number of quasi-one-dimensional systems, and it is also of interest to investigate the behavior of quantum field theories of relativistic Fermi fields.

Consider a quantum field theory of relativistic *fermions* in $1 + 1$ space-time dimensions. In $1 + 1$ dimensions fermi fields $\psi_a(x)$ are two-component spinors of the form denoted by $\psi_{R,a}$ the upper or *right* moving component of the fermion, and by $\psi_{L,a}$ the lower or *left* moving component of the fermion. In what follows we will consider the case in which there are N species of fermions labeled by an index $a = 1, \dots, N$. The free part of the Hamiltonian H_0 is

$$H_0 = \sum_{a=1}^N \int dx \left[\psi_{R\sigma}^\dagger(x) \left(-i \frac{\partial}{\partial x} \right) \psi_{R\sigma}(x) - \psi_{L\sigma}^\dagger(x) \left(-i \frac{\partial}{\partial x} \right) \psi_{L\sigma}(x) \right] \quad (1)$$

Here we have set the speed of the fermions (the speed of light) to unity.

The Fermi fields $\psi_{\alpha a}(x)$, with $\alpha = L, R$ and $a = 1, \dots, N$, satisfy equal-time canonical anticommutation relations

$$\{\psi_{\alpha a}^\dagger(x), \psi_{\beta b}(y)\} = \delta_{\alpha\beta} \delta_{ab} \delta(x - y). \quad (2)$$

The interaction part of the Hamiltonian, with coupling constant g , is

$$H_{\text{int}} = g \sum_{a,b=1}^N \int dx \psi_{R,\sigma}^\dagger(x) \psi_{L,\nu}^\dagger(x) \psi_{L,\nu}(x) \psi_{R,\sigma}(x) \quad (3)$$

The total Hamiltonian is $H = H_0 + H_{\text{int}}$. It is convenient to use a two-dimensional spinor notation

$$\psi_a = \begin{pmatrix} \psi_{R,a} \\ \psi_{L,a} \end{pmatrix} \quad (4)$$

and a set of two-dimensional Dirac γ matrices

$$\gamma_0 = \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \gamma_1 = i \sigma_2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \quad \gamma_5 = \gamma_0 \gamma_1 = -\sigma_3 = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \quad (5)$$

with the notation

$$\not{\partial} = \partial_\mu \gamma^\mu = \gamma_0 \partial_0 - \gamma_1 \partial_1 \quad (6)$$

where $\mu = 0, 1$ denote the indices of a 1 + 1-dimensional Minkowski space-time. Let us introduce the operators

$$\bar{\psi}\psi = \psi^\dagger \gamma_0 \psi \equiv \psi_R^\dagger \psi_L + \psi_L^\dagger \psi_R \quad (7)$$

and

$$\bar{\psi}\gamma_5\psi = \psi^\dagger \gamma_1 \psi \equiv \psi_R^\dagger \psi_L - \psi_L^\dagger \psi_R \quad (8)$$

Using this notation we can write the *normal ordered* Hamiltonian in the form

$$:H := \int dx : \psi_a^\dagger \gamma_5 \left(i \frac{\partial}{\partial x} \right) \psi_a : + : \frac{g}{4} [(\bar{\psi}_a \psi_a)^2 - (\bar{\psi}_a \gamma_5 \psi_a)^2] : \quad (9)$$

This continuum field theory is known as the $SU(N)$ (Chiral) Gross-Neveu model. In all the sections which follow below you have to use path-integral methods.

1. Find the Lagrangian density \mathcal{L} which yields the Hamiltonian and the anticommutation relations given above.
2. Derive an expression for the free fermion propagator in momentum space.
3. Derive the Feynman rules for a perturbation expansion of the fermion two-point function

$$S_{a,b}^{\alpha,\beta}(p) \quad (10)$$

(with $p = (p^0, p^1)$) in powers of the coupling constant g .

4. Derive an expression for the quantities listed below up to , and including, their second order corrections (*i.e.*, $O(g^2)$). Draw a Feynman diagram for each contribution. Check the cancellation of the vacuum diagrams. Give a consistent sign to each contribution.

- (a) the fermion two-point function
- (b) the effective coupling constant g . What correlation function should you consider?. Is it a connected Green function or a one-particle irreducible vertex function. Justify your answer.
Do not do the integrals!.

5. (a) Show that the Lagrangian that you found in i) is invariant under the *global* continuous symmetry

$$\psi_{\alpha a}(x) \rightarrow (e^{i\theta\gamma_5})_{\alpha\beta} \psi_{\beta a} \quad (11)$$

This invariance is known as chiral symmetry.

- (b) Find the transformation law obeyed by the operators $\hat{\Delta}_0 \equiv \bar{\psi}_a \psi_a$ and $\hat{\Delta}_5 \equiv \bar{\psi}_a i\gamma_5 \psi_a$ under this symmetry.

- (c) Give a physical interpretation of this symmetry. What is the meaning of the operators $\hat{\Delta}_0$ and $\hat{\Delta}_5$ in terms of the right and left moving components of the fermions?.
6. Now we add a chiral symmetry breaking term to the Lagrangian of the form

$$\mathcal{L}_{\text{chiral}} = H_0(x)\hat{\Delta}_0(x) + H_5(x)\hat{\Delta}_5(x) \quad (12)$$

where $H_0(x)$ and $H_5(x)$ are the symmetry breaking fields. Consider now the path integral for this problem in the presence of the symmetry breaking terms. Assume that the operators $\hat{\Delta}_0$ and $\hat{\Delta}_5$ have uniform expectation values given by $\bar{\Delta}_0$ and $\bar{\Delta}_5$ respectively. Find the transformation law obeyed by the symmetry breaking fields H_0 and H_5 under a global chiral transformation of the Fermi fields by an angle θ .

7. Derive a formal expression for the Effective Potential $U(\bar{\Delta}_0, \bar{\Delta}_5)$ in terms of a series expansion in powers of the expectation values. Relate the coefficients of this expansion in terms of vertex functions. What is the meaning of these vertex functions in terms of fermion operators?.
8. Find a perturbation theory expression for *all* the coefficients of the effective potential U to leading order (*i.e.*, the first non-vanishing term) in an expansion in terms of the coupling constant g . Do not do the integrals. Draw the Feynman diagram associated with each contribution.
Beware of the signs!!!!
9. Derive the chiral Ward Identity for the generating functional of the vertex functions for the operators $\bar{\Delta}_0$ and $\bar{\Delta}_5$.
10. Derive a Ward Identity which relates the vertex functions Γ_{00} and Γ_{55} in the limit $p \rightarrow 0$ ($p \equiv p_\mu$). Assume that, as $H_0 \rightarrow 0$ and $H_5 \rightarrow 0$, only $\bar{\Delta}_0$ has a non zero expectation value, *i.e.*, the chiral symmetry is spontaneously broken.
11. Is there a Goldstone boson in this system?. Justify your answer. Explain the significance of your answer to the original fermion problem. What does it say about its two-particle spectrum?. And about the one-particle spectrum?.

Comment: In this problem you were not required to do any integrals. Therefore you cannot determine if the chiral symmetry is spontaneously broken or not. In the next problem set we'll come back to this issue and do the integrals!.

2 Perturbative $SU(2)$ Yang-Mills Gauge Theory

In this part of the problem set you will consider the $SU(2)$ Yang-Mills gauge field $A_\mu(x)$ in a four dimensional Minkowski space-time. The Lagrangian density for

the $SU(2)$ Yang-Mills gauge theory in the Feynman-'t Hooft gauge is

$$\mathcal{L} = -\frac{1}{2}\text{tr}(F_{\mu\nu}F^{\mu\nu}) + \frac{\lambda}{2}\text{tr}(\partial_\mu A^\mu)^2 - \bar{\eta}_a \partial_\mu D_{ab}^\mu \eta_b \quad (13)$$

where A_μ is the gauge field and η_a and $\bar{\eta}$ are Faddeev-Popov ghosts. Please recall the following definitions (see my Physics 582/583 Lecture notes): is the covariant derivative in the fundamental representation,

$$D_{ab}^\mu = \partial^\mu \delta_{ab} + g f_{abc} A_c^\mu \quad (14)$$

is the covariant derivative in the *adjoint* representation, and

$$F_{\mu\nu} = t^a F_{\mu\nu}^a \quad (15)$$

is the field strength, where

$$F_{\mu\nu}^a = \partial_\mu A_\nu^a - \partial_\nu A_\mu^a + g f^{abc} A_\mu^b A_\nu^c \quad (16)$$

Here f^{abc} are the $SU(2)$ structure constants, *i.e.* $f^{abc} = \epsilon^{abc}$, and t^a (with $a = 1, 2, 3$) are three generators of $SU(2)$ which satisfy

$$[t^a, t^b] = i f^{abc} t^c, \quad \text{tr}(t^a t^b) = \frac{1}{2} \delta_{ab} \quad (17)$$

For $SU(2)$ in the fundamental (spinor) representation, the generators are simply related to the three 2×2 Hermitian Pauli matrices $\{\sigma^a\}$:

$$t^a = \frac{1}{2} \sigma^a \quad (18)$$

1. Write down the Feynman propagators in momentum space for the gluons (gauge fields) A_μ^a and for the Faddeev-Popov ghosts η^a and $\bar{\eta}^a$.
2. Use the Lagrangian density given above to derive the Feynman rules for this theory in position space. Draw diagrams as you feel may be necessary.
3. Use the (Feynman gauge) Feynman rules derived in the last section, to find the one-loop correction in a perturbation expansion in powers of the gauge coupling constant g for
 - (a) the gauge field propagator
 - (b) the three and four gauge field vertices

in this gauge. **Do not do the integrals.** Draw a Feynman diagram for each contribution.

3 Renormalization of the $O(N)$ scalar ϕ^4 theory

Consider the ϕ^4 theory in a regime in which the renormalized mass is non-zero for a scalar field ϕ_a where $a = 1, \dots, N$, with an $O(N)$ -invariant Lagrangian density in D Euclidean dimensions of the form

$$\mathcal{L} = \frac{1}{2} (\nabla_\mu \phi_a)^2 + \frac{m_0^2}{2} \phi_a^2 + \frac{\lambda}{4!} (\phi_a^2)^2 \quad (19)$$

In the symmetric theory, *i.e.* in the absence of spontaneous symmetry breaking, the two-point and four-point 1PI vertex functions $\Gamma_{ab}^{(2)}(p)$ and $\Gamma_{abcd}^{(4)}(p_1, \dots, p_4)$ take the symmetric form

$$\Gamma_{ab}^{(2)}(p) = \delta_{ab} \bar{\Gamma}^{(2)}(p) \quad (20)$$

and

$$\Gamma_{abcd}^{(4)}(p_1, \dots, p_4) = S_{abcd} \bar{\Gamma}^{(4)}(p_1, \dots, p_4) \quad (21)$$

where

$$S_{abcd} = \frac{1}{3} (\delta_{ab} \delta_{cd} + \delta_{ac} \delta_{bd} + \delta_{ad} \delta_{bc}) \quad (22)$$

1. Find all the contributions to $\Gamma^{(2)}$ and $\Gamma^{(4)}$ to **one loop order** for this N -component theory. Make sure to write down and to draw all Feynman diagrams and their associated analytic expressions. Write your results in terms of the integrals discussed in class (for the one-component theory). Do not do the integrals. Derive the explicit dependence of each diagram on the number of components N .
2. Define a set of *renormalization conditions*, at zero external momentum, for the vertex functions $\bar{\Gamma}^{(2)}(p)$ and $\bar{\Gamma}^{(4)}(p_1, \dots, p_4)$ in the symmetric massive theory.
3. Determine m_0^2 , and λ in terms of the renormalized mass μ and coupling constant g at fixed momentum cutoff Λ , **to one loop order**. Express your answers in terms of the integrals defined in class. **Do not do the integrals!**
4. Show that the renormalization conditions of part 2) and the renormalization constants obtained in part 3) yields finite vertex functions at arbitrary values of the external momentum, **to one loop order** in perturbation theory.

Useful identities:

$$\sum_c S_{abcc} = \frac{N+2}{3} \delta_{ab}$$

$$\sum_{ij} S_{abij} S_{ijcd} = \frac{2}{3} S_{abcd} + \frac{N+2}{3} \delta_{ab} \delta_{cd}$$