Physics 582, Fall Semester 2008
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Problem Set No. 1: Classical Field Theory
Due Date: September 12, 2008

1 The Landau Theory of Phase Transitions as a Classical Field Theory

In the Landau-Ginzburg approach to the theory of phase transitions, the thermodynamic properties of a one-component classical ferromagnet in thermal equilibrium are described by a *free energy functional* of an order-parameter field $\phi(\vec{x})$ (the local magnetization). This functional contains, in addition to gradient terms, contributions proprtional to various powers of the local order parameter. Under some circumstances the coefficient λ of the ϕ^4 term of the energy functional may become negative. This is what happens if the local magnetic moments have spin-1 rather the spin-$\frac{1}{2}$. In this case, we have to include, in the energy functional, a term with a higher power of ϕ (such as ϕ^6) in order to insure the thermodynamic stability of the system.

The (free) energy density $\mathcal{E}$ for this system has the form

$$\mathcal{E} = \frac{1}{2}\left(\vec{\bigtriangledown}\phi(\vec{x})\right)^2 + U(\phi(\vec{x}))$$

where the potential $U(\phi(\vec{x}))$ is

$$U(\phi(\vec{x})) = \frac{m_0^2}{2}\phi^2(\vec{x}) + \frac{\lambda_4}{4!}\phi^4(\vec{x}) + \frac{\lambda_6}{6!}\phi^6(\vec{x}).$$

with $m_0^2 = a(T - T_0)$ and $\lambda_4 < 0$, $\lambda_6 > 0$.

1. Use a variational principle to derive the saddle-point equations (*i.e.,* the Landau-Ginzburg equations) for this system.

2. Plot the potential $U(\phi)$ for a constant field $\bar{\phi}$ for $\lambda_4 < 0$ (and fixed) at several temperatures. Show that, as the temperature T is lowered, there exists a temperature $T^* > T_0$ at which the state with lowest energy has $\langle\phi\rangle \neq 0$ (for fixed $\lambda_4 < 0$ and $\lambda_6 > 0$). Plot the qualitative behavior of $\langle\phi\rangle$ as a function of T. Is this a continuous function? Is this a first order or a second order transition? Find the value of the energy of the system in the ordered state.

3. Consider now the case $\lambda_4 > 0$ and show that the transition now takes place at T_0. Plot the qualitative behavior of $\langle\phi\rangle$ as a function of T for this case. Is this a continuous function? Is this a first order or a second order transition?

4. Collect your results of the previous sections in the form of a plot of λ_4 as a function of $T - T_0$. Indicate on the graph in which areas is the system ordered and in which ones it is disordered. Indicate where is the transition first order and where it is second order. Find an analytic expression for the phase boundary, the curve that separates the ordered and disordered states.

2 Scalar Electrodynamics

The dynamics of a *charged* (complex) scalar field $\phi(x)$ coupled to the electromagnetic field $A_\mu(x)$ is governed by the Lagrangian density $\mathcal{L}$

$$\mathcal{L} = \frac{1}{2}\left(D_\mu\phi(x)\right)^*\left(D^\mu\phi(x)\right) - \frac{m_0^2}{2}|\phi(x)|^2 - \frac{\lambda}{4!}\left(|\phi(x)|^2\right)^2 - \frac{1}{4}F^{\mu\nu}F_{\mu\nu}$$

where D_μ is the *covariant derivative*

$$D_\mu \equiv \partial_\mu + ieA_\mu$$

e is the electric charge and * denotes complex conjugation.

1. Show that this Lagrangian density is invariant under the local gauge transformations

$$\begin{aligned}\phi'(x) &= \phi(x)\, e^{-ie\Lambda(x)}\\ \phi'^*(x) &= \phi^*(x)\, e^{+ie\Lambda(x)}\\ A'_\mu(x) &= A_\mu(x) + \partial_\mu\Lambda(x)\end{aligned}$$

2. Derive the classical equations of motion in a manifestly relativistically covariant form.

3. Find the Hamiltonian density for this system.

4. Write the complex field $\phi(x)$ in its polar components

$$\phi(x) = \rho(x)\, e^{i\theta(x)}$$

and find the equations of motion obeyed by the *real* fields ρ and θ. Write these equations of motion in the gauge $\theta = 0$, known as the London or Unitary gauge. Find the Lagrangian for the field ρ.

5. Show that, if $m_0^2 < 0$, the equation of motion for the field ρ in the London gauge (derived above) has a solution with $\rho = \bar{\rho} =$ constant > 0. Freeze the field ρ at the value $\bar{\rho}$ and find the effective Lagrangian for the remaining degrees of freedom A_μ. Show that this Lagrangian has a term which is *quadratic* in A_μ and calculate its coefficient. By solving the equations

of motion for A_μ (derived from this effective Lagrangian) show that the coefficient of this quadratic term can be interpreted as a *photon mass*. Note: This phenomenon is known as the Meissner effect. This theory represents a system in a superconducting state.

3 The Dirac Equation

1. Use the Dirac equation to show that the 4-current $j^\mu = \bar{\psi}\gamma^\mu\psi$ is conserved.

2. Show that if ψ is a 4-spinor which satisfies the Dirac equation, then ψ also satisfies the Klein-Gordon equation.

3. Verify that the following identities hold

   (a)
   $$\not{A}\not{B} = A \cdot B - i\sigma_{\mu\nu}A^\mu B^\nu$$
   where A^μ and B^ν are two arbitrary 4-vectors.

   (b)
   $$tr\not{A}\not{B} = 4\; A \cdot B$$

   (c)
   $$\gamma^\lambda\gamma^\mu\gamma_\lambda = -2\gamma^\mu$$

4 Transformation Properties of Field Bilinears in the Dirac Theory

In this problem you will consider again the Dirac Theory and study the transformation properties of its physical observable under Lorentz transformations. Let

$$x'^\mu = \Lambda^\mu_\nu\; x^\nu$$

be a general Lorentz transformation, and $S(\Lambda)$ be the induced transformation for the Dirac spinors $\psi_a(x)$ (with $a = 1, \ldots, 4$):

$$\psi'_a(x') = S(\Lambda)_{ab}\; \psi_b(x)$$

Verify that the following Dirac bilinears listed below obey the following transformation laws:

1.
$$\bar{\psi}'(x'\;)\; \psi'(x'\;) = \bar{\psi}(x)\; \psi(x)$$

2.
$$\bar{\psi}'(x'\;)\; \gamma_5\; \psi'(x'\;) = \det\Lambda \bar{\psi}(x)\; \gamma_5\; \psi(x)$$

3.

$$\bar{\psi}'(x')\,\gamma^\mu\,\psi'(x') = \Lambda^\mu_\nu\,\bar{\psi}(x)\,\gamma^\nu\,\psi(x)$$

4.

$$\bar{\psi}'(x')\,\gamma_5\gamma^\mu\,\psi'(x') = \det\Lambda\;\Lambda^\mu_\nu\,\bar{\psi}(x)\,\gamma_5\gamma^\nu\,\psi(x)$$

5.

$$\bar{\psi}'(x')\,\sigma^{\mu\nu}\,\psi'(x') = \Lambda^\mu_\alpha\,\Lambda^\nu_\beta\,\bar{\psi}(x)\,\sigma^{\alpha\beta}\,\psi(x)$$

where Λ^μ_ν is a Lorentz transformation and $\det\Lambda$ is its determinant.

5 Chiral Symmetry

Let us, once again, consider the Dirac equation

$$\left(i\partial\!\!\!/ - m\right)\psi = 0$$

but, this time, in the *Chiral* representation for the Dirac γ-matrices, *i.e.,*

$$\begin{aligned}
\gamma^0 &= -\sigma^1 \otimes I = \begin{pmatrix} 0 & -I \\ -I & 0 \end{pmatrix} \\
\vec{\gamma} &= i\sigma^2 \otimes \vec{\sigma} = \begin{pmatrix} 0 & \vec{\sigma} \\ -\vec{\sigma} & 0 \end{pmatrix} \\
\gamma_5 &= \gamma^5 = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix} \\
\sigma^{0i} &= i\begin{pmatrix} \sigma^i & 0 \\ 0 & -\sigma^i \end{pmatrix} \\
\sigma^{ij} &= \epsilon^{ijk}\begin{pmatrix} \sigma^k & 0 \\ 0 & \sigma^k \end{pmatrix}
\end{aligned}$$

where σ^i are the three Pauli matrices and I is the 2×2 identity matrix. Recall the definition of the matrix $\gamma_5 = i\gamma^0\gamma^1\gamma^2\gamma^3$.

1. Using the Dirac matrices in the Chiral representation, write down the Dirac equation in terms of the 2-spinors ϕ and χ, where

$$\psi = \begin{pmatrix} \phi \\ \chi \end{pmatrix}$$

2. Show that if the excitations have zero mass (*i.e.,* $m = 0$) the Dirac equation, written in the chiral basis, decouples into two 2×2 equations. Find the plane wave solutions of these equations and calculate their dispersion law (*i.e.,* energy-momentum relation). Assign a chirality (γ_5) quantum number to each solution.

3. Consider now the *chiral* transformation (CT)

$$\psi' = e^{i\gamma_5\theta}\ \psi$$

(a) Find how do the 2-spinors ϕ and χ transform under a CT.

(b) Find how does $\bar{\psi}$ transform under a CT.

(c) Find the transformation laws under a CT of the bilinears $\bar{\psi}\psi$ and $\bar{\psi}\gamma^\mu\psi$.

(d) Is the Dirac equation covariant under a CT if $m \neq 0$? Find the form of the Dirac equation , in terms of 4-spinors ψ, after a CT with angle θ has been carried out. What new terms do you find?