

Physics 581: Quantum Mechanics II
Department of Physics, UIUC
Spring Semester 2007
Professor Eduardo Fradkin
Problem Set No. 6/ Final Exam:
The Dirac equation
Due Date: 5/11/2007

1 Dirac Theory for a Relativistic Electron in a Potential Well

In this problem you will consider the problem of localizing a Dirac fermion, *e.g.* an electron, by means of an external electrostatic potential well. To make the matters simpler, we will assume that the system is restricted to move on a line (say the axis $x_3 \equiv x$) and that, as a result, the problem is effectively one-dimensional. In what follows assume that the fields do not depend on the other coordinates.

1. Find the Hamiltonian for the problem of a Dirac field coupled to a static external electrostatic field $\vec{E} = -\vec{\nabla}\Phi$, for the case of this one-dimensional geometry.
2. Find the single-particle states for the Dirac theory in the presence of the potential well

$$\Phi(x) = \begin{cases} -U_0, & \text{for } |x| \leq \frac{a}{2} \\ 0, & \text{otherwise.} \end{cases} \quad (1)$$

where $U_0 > 0$. Be careful to include both extended and bound states and to discuss the behavior of these states as a function of the depth U_0 and of the mass m . What boundary conditions should you impose at $x = \pm a/2$? Explain how do these boundary conditions differ from the case of the non-relativistic Schrödinger equation for the same system and why.

3. Construct the ground state and the excited state with lowest energy for the various regimes that you found in section (b) and give their quantum numbers. Derive an expansion of the ground state *in the presence* of the potential in terms of the eigenstates of the same theory *without* the potential. Give a physical interpretation of the various terms of this expansion.

2 The Klein-Gordon Hydrogen Atom

It was shown in class that the Klein-Gordon equation for a spin zero particle of charge $-e$ coupled to an electromagnetic field is

$$\left[\frac{1}{c^2} \left(i\hbar \frac{\partial}{\partial t} - e\Phi(\vec{x}, t) \right)^2 - \left(\frac{\hbar}{i} \vec{\nabla} - \frac{e}{c} \vec{A}(\vec{x}, t) \right)^2 - m^2 c^2 \right] \phi(\vec{x}, t) = 0 \quad (2)$$

where $\phi(\vec{x}, t)$ is the Klein-Gordon field for a charged spin zero boson, *e. g.* a negatively charged pion π^- of mass m . Here $\Phi(\vec{x}, t)$ is the electromagnetic scalar potential and $\vec{A}(\vec{x}, t)$ is the electromagnetic vector potential. In this problem you will be looking at the case in which the pion is coupled only to the electrostatic field of a very heavy nucleus of charge Ze :

$$\Phi(\vec{x}) = \frac{-Ze}{|\vec{x}|}, \quad \vec{A} = 0 \quad (3)$$

In this problem you will find the spectrum of energy eigenstates and their associated “wave functions” $\phi(\vec{x})$ for this relativistic atom, where as usual we will set

$$\phi(\vec{x}, t) = e^{-iEt/\hbar} \phi(\vec{x}) \quad (4)$$

1. Derive the equation satisfied by the eigenstates $\phi(\vec{x})$ with energy eigenvalue E . Show that for this central potential $\phi(\vec{x})$ can be factored into an eigenstate of total angular momentum $Y_{\ell, m}(\theta, \varphi)$ and a radial wave function $R(r)$ which satisfies the eigenvalue equation

$$\left[\left(\frac{E^2}{c^2} - m^2 c^2 \right) + \hbar^2 \left(\frac{1}{r} \frac{\partial^2}{\partial r^2} r - \frac{\ell(\ell+1) - Z^2 \alpha^2}{r^2} \right) + \frac{2Ze^2 E}{r c^2} \right] R(r) = 0 \quad (5)$$

where $\alpha = \frac{e^2}{\hbar c} \simeq \frac{1}{137}$ is the fine structure constant.

2. Find an explicit formula for the spectrum of bound states $E_{n, \ell}$ of Eq. 5 in terms of the mass of the pion m , the speed of light c , the charge e , the atomic number Z , the fine structure constant α and the quantum numbers n and ℓ . (Hint: You may want to bring this problem into the familiar form of the radial Schrödinger Equation for a particle of an effective mass m' , by defining a suitable angular momentum ℓ' , an effective radial number n' and an effective energy E' .) What is the relation between ℓ' , ℓ and the fine structure constant. can you construct these solutions for all integer values of ℓ or are there restrictions? Discuss physically the nature of these restrictions.
3. What happened to the accidental degeneracy of the non-relativistic Hydrogen spectrum? Does it survive in the relativistic case? Expand the energy eigenvalues to order α^4 (included). Compare your results with the non-relativistic spectrum.

3 The Dirac Equation and γ matrices

In this exercise you will consider the Dirac Equation with the Dirac γ matrices in the Dirac representation (which we used in class):

$$\begin{aligned}
 \gamma^0 &= \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix} \\
 \gamma^i &= \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix} \\
 \gamma_5 &= \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix} \\
 \sigma^{0i} &= i \begin{pmatrix} 0 & \sigma^i \\ \sigma^i & 0 \end{pmatrix} \\
 \sigma^{ij} &= \epsilon_{ijk} \begin{pmatrix} \sigma^k & 0 \\ 0 & \sigma^k \end{pmatrix}
 \end{aligned} \tag{6}$$

where $i, j, k = 1, 2, 3$, $\gamma_5 = i\gamma^0\gamma^1\gamma^2\gamma^3$, and

$$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{7}$$

1. Use the Dirac equation to show that the 4-current $j^\mu = \bar{\psi}\gamma^\mu\psi$ is conserved.
2. Show that if ψ is a 4-spinor which satisfies the Dirac equation, then ψ also satisfies the Klein-Gordon equation.
3. Verify that the following identities hold

$$\text{(a)} \quad \not{A}\not{B} = A \cdot B - i\sigma_{\mu\nu}A^\mu B^\nu$$

where A^μ and B^ν are two arbitrary 4-vectors.

$$\text{(b)} \quad \text{tr}\not{A}\not{B} = 4 A \cdot B$$

$$\text{(c)} \quad \gamma^\lambda\gamma^\mu\gamma_\lambda = -2\gamma^\mu$$

4. Let

$$x'^\mu = \Lambda_\nu^\mu x^\nu$$

be a general Lorentz transformation, $\det\Lambda$ be its determinant, and $S(\Lambda)$ be the induced transformation for the Dirac spinors $\psi_a(x)$ (with $a = 1, \dots, 4$):

$$\psi'_a(x') = S(\Lambda)_{ab} \psi_b(x)$$

Show that the Dirac bilinears listed below transform as follows under a Lorentz transformation:

(a) scalar:

$$\bar{\psi}'(x') \psi'(x') = \bar{\psi}(x) \psi(x)$$

(b) pseudoscalar:

$$\bar{\psi}'(x') \gamma_5 \psi'(x') = \det \Lambda \bar{\psi}(x) \gamma_5 \psi(x)$$

(c) vector:

$$\bar{\psi}'(x') \gamma^\mu \psi'(x') = \Lambda_\nu^\mu \bar{\psi}(x) \gamma^\nu \psi(x)$$

(d) pseudovector:

$$\bar{\psi}'(x') \gamma_5 \gamma^\mu \psi'(x') = \det \Lambda \Lambda_\nu^\mu \bar{\psi}(x) \gamma_5 \gamma^\nu \psi(x)$$

(e) tensor:

$$\bar{\psi}'(x') \sigma^{\mu\nu} \psi'(x') = \Lambda_\alpha^\mu \Lambda_\beta^\nu \bar{\psi}(x) \sigma^{\alpha\beta} \psi(x)$$

4 Chiral Symmetry

Consider the Dirac equation in the *chiral* representation for the Dirac γ -matrices,

$$\begin{aligned}\gamma^0 &= \begin{pmatrix} 0 & -I \\ -I & 0 \end{pmatrix} \\ \vec{\gamma} &= \begin{pmatrix} 0 & \vec{\sigma} \\ -\vec{\sigma} & 0 \end{pmatrix} \\ \gamma_5 &= \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix} \\ \sigma^{0i} &= i \begin{pmatrix} \sigma^i & 0 \\ 0 & -\sigma^i \end{pmatrix} \\ \sigma^{ij} &= \epsilon^{ijk} \begin{pmatrix} \sigma^k & 0 \\ 0 & \sigma^k \end{pmatrix}\end{aligned}$$

where σ^i are the three Pauli matrices and I is the 2×2 identity matrix. Recall the definition of the matrix $\gamma_5 = i\gamma^0\gamma^1\gamma^2\gamma^3$.

- Using the Dirac matrices in the Chiral representation, write down the Dirac equation in terms of the 2-spinors ϕ and χ , where

$$\psi = \begin{pmatrix} \phi \\ \chi \end{pmatrix}$$

- Show that if the excitations have zero mass (*i.e.*, $m = 0$) the Dirac equation, written in the chiral basis, decouples into two 2×2 equations. Find the plane wave solutions of these equations and calculate their dispersion law (*i.e.*, energy-momentum relation). Assign a chirality (γ_5) quantum number to each solution.
- Consider now the *chiral* transformation (CT)

$$\psi' = e^{i\gamma_5\theta} \psi$$

- Find how do the 2-spinors ϕ and χ transform under a CT.
- Find how does $\bar{\psi}$ transform under a CT.
- Find the transformation laws under a CT of the bilinears $\bar{\psi}\psi$ and $\bar{\psi}\gamma^\mu\psi$.
- Is the Dirac equation covariant under a CT if $m \neq 0$? Find the form of the Dirac equation, in terms of 4-spinors ψ , after a CT with angle θ has been carried out. What new terms do you find?