Physics 581: Quantum Mechanics II
Department of Physics, UIUC
Spring Semester 2007
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Problem Set No. 5:
Matter and Electromagnetic Radiation
Due Date: 4/30/2007

1 Quantization of the Electromagnetic Field

Consider the free electromagnetic field in the Coulomb gauge, $\vec{\bigtriangledown} \cdot \vec{A} = 0$. The vector potential $\vec{A}(\vec{r}, t)$ in the has the mode expansion

$$\vec{A}(\vec{r}) = \int \frac{d^3k \, \sqrt{4\pi}\hbar c}{2\pi^3 2E(\vec{k})} \sum_{\lambda=1,2} \vec{\epsilon}(\vec{k}, \lambda) \left[\hat{a}(\vec{k}, \lambda) \; e^{-i\vec{k}\cdot\vec{r}} + \hat{a}(\vec{k}, \lambda)^\dagger \; e^{i\vec{k}\cdot\vec{r}} \right] \tag{1}$$

where $E(\vec{k}) = \hbar c|\vec{k}$. The operators $\hat{a}(\vec{k}, \lambda)$ and $\hat{a}(\vec{k}, \lambda)^\dagger$ are the photon destruction and creation operators which satisfy the equal time commutation relations

$$[\hat{a}(\vec{k}, \lambda), \hat{a}(\vec{k}', \lambda')] = [\hat{a}(\vec{k}, \lambda)^\dagger, \hat{a}(\vec{k}', \lambda')^\dagger] = 0 \qquad [\hat{a}(\vec{k}, \lambda), \hat{a}(\vec{k}', \lambda')^\dagger] = \delta_{\lambda,\lambda'} \delta(\vec{k} - \vec{k}') \tag{2}$$

The the (linear) polarization unit vectors $\vec{\epsilon}\,(\vec{k}, \lambda)$ were defined in class. Together with the unit vector $\hat{k} = \vec{k}/|\vec{k}|$, they form an orthonormal and complete basis:

$$\vec{\epsilon}\,(\vec{k}, \lambda) \cdot \vec{\epsilon}\,(\vec{k}, \lambda') = \delta_{\lambda\lambda'}, \quad \vec{\epsilon}\,(\vec{k}, \lambda) \cdot \vec{k} = 0, \quad \sum_{\lambda=1,2} \epsilon_i(\vec{k}, \lambda)\epsilon_j(\vec{k}, \lambda) + \frac{k_i k_j}{|\vec{k}|^2} = \delta_{ij} \tag{3}$$

with $i, j = 1, 2, 3$ (or x, y, z).

1. Show that the classical energy and momentum of the electromagnetic field

$$H = \int d^3r \frac{1}{8\pi} \left(\vec{E}^2 + \vec{B}^2 \right) \qquad \text{and} \qquad \vec{P} = \frac{1}{4\pi c} \int d^3r \; \vec{E} \times \vec{B} \tag{4}$$

become, in the quantum theory, the operators

$$\hat{H} = \int d^3k \sum_\lambda E(\vec{k}) \; \hat{a}(\vec{k}, \lambda)^\dagger \; \hat{a}(\vec{k}, \lambda) \tag{5}$$

$$\vec{P} = \int d^3k \sum_\lambda \hbar\vec{k} \; \hat{a}(\vec{k}, \lambda)^\dagger \; \hat{a}(\vec{k}, \lambda) \tag{6}$$

2. All the operators written below are in the **Heisenberg representation** generated by the Hamiltonian of Eq. 6. Compute the following equal-time commutation relations $[\hat{A}_i(\vec{r}, t), \hat{A}_j(\vec{r}', t)]$, $[\hat{E}_i(\vec{r}, t), \hat{E}_j(\vec{r}', t)]$, $[\hat{A}_i(\vec{r}, t), \hat{E}_j(\vec{r}', t)]$ and $[\hat{E}_i(\vec{r}, t), \hat{B}_j(\vec{r}', t)]$.

3. Do $\hat{E}_i(\vec{r},t)$ and/or $\hat{B}_j(\vec{r},t)$ commute with the *total* photon number operator $\hat{N}$?

$$\hat{N} = \int d^3k \sum_\lambda \hat{a}(\vec{k},\lambda)^\dagger \, \hat{a}(\vec{k},\lambda) \tag{7}$$

Interpret this result physically.

4. Use the Heisenberg equations of motion which, for a generic operator $\hat{Q}$ are

$$i\hbar \frac{d}{dt}\hat{Q} = [\hat{Q}, \hat{H}] \tag{8}$$

to compute

$$i\hbar \frac{d}{dt}\vec{E}(\vec{r},t) = [\vec{E}(\vec{r},t), \hat{H}] \tag{9}$$

and

$$i\hbar \frac{d}{dt}\vec{B}(\vec{r},t) = [\vec{E}(\vec{r},t), \hat{H}] \tag{10}$$

Compare your results with Maxwell's Equations. Did you get all of them?

5. Show that

$$\begin{aligned}
\left[\vec{\nabla} \cdot \vec{E}(\vec{r},t), \hat{E}_i(\vec{r}\,',t)\right] &= 0 \\
\left[\vec{\nabla} \cdot \vec{E}(\vec{r},t), \hat{B}_i(\vec{r}\,',t)\right] &= 0 \\
\left[\vec{\nabla} \cdot \vec{E}(\vec{r},t), \hat{H}\right] &= 0
\end{aligned} \tag{11}$$

Do the same calculation for the operator $\vec{\nabla} \cdot \vec{B}(\vec{r},t)$. Assume that, *at some time* t_0, we have a state $|\Psi(t_0)\rangle$ of the electromagnetic field with the properties, $\vec{\nabla} \cdot \vec{E}(\vec{r}\ )|\Psi(t_0)\rangle = 0$ and $\vec{\nabla} \cdot \vec{B}(\vec{r}\ )|\Psi(t_0)\rangle = 0$. Will this *constraint* remain true at all times $t \geq t_0$? Explain your answer. How is this property related to the *classical* Maxwell's equations?

6. Use creation and annihilation operator methods to construct the *correctly normalized* one photon states $|\vec{p}, R\rangle$ and $|\vec{p}, L\rangle$, where R, L stand for *right and left* circular polarization. Use the expression of the Hamiltonian, total momentum an intrinsic angular momentum to show that these are eigenstates, and to determine the values of these observables in these photon states.

2 Scattering of Light

Consider the problem of scattering of light by a charged system, such as an atom. Both the e. m. field and the atom will be considered quantum mechanically. We will denote the states of the system by the kets $|n; N(\vec{p},\nu)\rangle$, where $|n\rangle$ is an eigenstate of the charged system, and $|N(\vec{p},\nu)\rangle$ is an eigenstate of the e. m.

field labelled by its photon occupation numbers. Let us consider the following scattering processes between initial and final states $|I\rangle$ and $|F\rangle$ of the form

$$|I\rangle = |0; N(\vec{p},\nu), N(\vec{p}\,',\nu') = 0\rangle \to |F\rangle = |n; N(\vec{p},\nu) - 1, N(\vec{p}\,',\nu') = 1\rangle \quad (12)$$

This process involves two photons and, as such, it can either occur through the term $-\frac{e}{c}\int d^3r \vec{j}(\vec{r}) \cdot \vec{A}(\vec{r})$ to *second order in perturbation theory* or through the (diamagnetic) term $\frac{e^2}{2mc^2}\int d^3r \rho(\vec{r})\vec{A}^2(\vec{r})$ to first order in perturbation theory. In this problem you will look at the effects of this second term. Thus, the interaction term is

$$H^{(2)}_{\rm int} = \frac{e^2}{2mc^2}\int d^3r \rho(\vec{r})\vec{A}^2(\vec{r}) \quad (13)$$

where $\rho(\vec{r})$ is the density operator (defined in class).

1. Find the form of the correctly normalized incoming and outgoing photon states.
2. Use creation and annihilation operators to compute the matrix element
$$\langle N(\vec{p},\nu) - 1, N(\vec{p}\,',\nu') = 1|\vec{A}^2(\vec{r})|N(\vec{p},\nu), N(\vec{p}\,',\nu') = 0\rangle \quad (14)$$
3. Use the result you just derived to show that the matrix element of interest is
$$\begin{aligned}\langle n; N(\vec{p},\nu) - 1, N(\vec{p}\,',\nu') = 1|\frac{e^2}{2mc^2}\int d^3r \rho(\vec{r})\vec{A}^2(\vec{r})|0; N(\vec{p},\nu), N(\vec{p}\,',\nu') = 0\rangle \\ = r_0 \frac{2\pi\hbar c^2}{\sqrt{\omega\omega'}}\frac{\sqrt{N(\vec{p},\nu)}}{V}\vec{\epsilon}(\vec{p},\nu)\cdot\vec{\epsilon}(\vec{p}\,',\nu')\langle n|\int d^3r \rho(\vec{r})e^{i(\vec{p}-\vec{p}\,')\cdot\vec{r}}|0\rangle \quad (15)\end{aligned}$$
where $r_0 = \frac{e^2}{mc^2} = 2.8 \times 10^{-13}$ cm, is the (so-called) classical radius of the electron.
4. Use the previous result to show that the *differential cross section* $\frac{d\sigma_{0\to n}}{d\Omega}$ for this process is
$$\frac{d\sigma_{0\to n}}{d\Omega} = r_0^2 \frac{\omega}{\omega'}|\vec{\epsilon}(\vec{p},\nu)\cdot\vec{\epsilon}(\vec{p}\,',\nu')|^2|\langle n|\rho(\vec{p}-\vec{p}\,')|0\rangle|^2 \quad (16)$$
Note: Recall that the differential cross section is the ration of the differential rate (for the same transition) and the incoming photon flux $cN(\vec{p},\nu)/V$, where V is the volume.
5. Specialize the result of the previous section for the case of scattering of light by a single free electron whose initial state is $|0\rangle = |\hbar\vec{q}_0\rangle$ and final state $|n\rangle = |\hbar\vec{q}_n\rangle$ and derive an expression of the differential cross section for this case (this is usually called the *Thomson cross section*). Explain why this process is allowed.

3 Spontaneous Emission

A hydrogen atom is *at rest* (*i. e.*, the center of mass has zero momentum), in the state $|n, m, l\rangle = |2, 1, 1\rangle$, in a large box of volume V.

1. Calculate the details of the spontaneous emission from the atom as it decays to the ground state. In your calculations use the appropriate form of the dipole approximation for this problem.

2. Compare the lifetime of the state $|2, 1, 1\rangle$ here with the lifetime when the proton is *anchored* in space.

3. If in the final state, the photon moves along the (positive) z axis (which we take to be the quantization axis for the orbital angular momentum of the electron), what is the polarization of this photon? Explain your answer and make sure to use conservation laws. Also, is the total momentum conserved?