

# 1 Quantization of the Electromagnetic Field

## 1.

The electric and magnetic field are given by the vector potential:

$$\begin{aligned}\vec{E} &= -\frac{1}{c} \frac{\partial \vec{A}}{\partial t} \\ \vec{B} &= \nabla \times \vec{A}\end{aligned}$$

The electromagnetic energy reads:

$$H = \int d^3\vec{r} \frac{1}{8\pi} \left[ \frac{1}{c^2} \left( \frac{\partial \vec{A}}{\partial t} \right)^2 + (\nabla \times \vec{A})^2 \right]$$

The time dependence of the vector potential can be restored by applying the rule:

$$\hat{a}(\vec{k}, \lambda) \rightarrow \hat{a}(\vec{k}, \lambda) e^{-iE(\vec{k})t/\hbar}$$

where  $E(\vec{k}) = \hbar c |\vec{k}|$ . The time dependent vector potential is given by:

$$\vec{A}(\vec{r}, t) = \int \frac{d^3\vec{k} \sqrt{4\pi\hbar c}}{\sqrt{(2\pi)^3 2E(\vec{k})}} \sum_{\lambda=1,2} \vec{\epsilon}(\vec{k}, \lambda) \left[ \hat{a}(\vec{k}, \lambda) e^{i\vec{k}\cdot\vec{r} - iE(\vec{k})t/\hbar} + \hat{a}(\vec{k}, \lambda)^\dagger e^{-i\vec{k}\cdot\vec{r} + iE(\vec{k})t/\hbar} \right] \quad (1)$$

NOTE: There are a few typos in the given expression. Consult the Lecture Notes, page 115.

To simplify the derivation a bit we are going to introduce the notation:

$$\begin{aligned}\vec{A}(\vec{k}, t) &= \frac{\sqrt{4\pi\hbar c}}{\sqrt{2E(\vec{k})}} \sum_{\lambda=1,2} \left[ \vec{\epsilon}(-\vec{k}, \lambda) \hat{a}(-\vec{k}, \lambda) e^{-iE(\vec{k})t/\hbar} + \vec{\epsilon}(\vec{k}, \lambda) \hat{a}(\vec{k}, \lambda)^\dagger e^{iE(\vec{k})t/\hbar} \right] \\ \vec{A}(\vec{r}, t) &= \int \frac{d^3\vec{k}}{(2\pi)^{3/2}} \vec{A}(\vec{k}, t) e^{-i\vec{k}\cdot\vec{r}}\end{aligned} \quad (2)$$

It is not difficult to show that after carrying out the Fourier transform the energy will take the form:

$$H = \frac{1}{8\pi} \int d^3\vec{k} \left[ \frac{1}{c} \frac{\partial}{\partial t} \vec{A}(\vec{k}, t) \cdot \frac{1}{c} \frac{\partial}{\partial t} \vec{A}(-\vec{k}, t) + (\vec{k} \times \vec{A}(\vec{k}, t)) \cdot (\vec{k} \times \vec{A}(-\vec{k}, t)) \right]$$

This is obtained by just performing the spacial integration and eliminating the  $\delta(\vec{k} + \vec{k}')$  which will emerge. For the magnetic field term we can do the algebra:

$$\begin{aligned}(\vec{k} \times \vec{A}(\vec{k}, t)) \cdot (\vec{k} \times \vec{A}(-\vec{k}, t)) &= -\vec{k} \cdot \left( (\vec{k} \times \vec{A}(\vec{k}, t)) \times \vec{A}(-\vec{k}, t) \right) \\ &= -\vec{k} \cdot \left( \vec{A}(\vec{k}, t) (\vec{k} \cdot \vec{A}(-\vec{k}, t)) - (\vec{A}(\vec{k}, t) \cdot \vec{A}(-\vec{k}, t)) \vec{k} \right) \\ &= \vec{k}^2 (\vec{A}(\vec{k}, t) \cdot \vec{A}(-\vec{k}, t))\end{aligned}$$

In the first line we used the identity  $\vec{a} \cdot (\vec{b} \times \vec{c}) = -\vec{b} \cdot (\vec{a} \times \vec{c})$  and not the  $\vec{a} \cdot (\vec{b} \times \vec{c}) = \vec{b} \cdot (\vec{c} \times \vec{a})$  because  $\vec{A}(\vec{k}, t)$  and  $\vec{A}(-\vec{k}, t)$  are operators and we cannot permute them. Accordingly in the second line we used the identity  $(\vec{b} \times \vec{c}) \times \vec{a} = \vec{c}(\vec{a} \cdot \vec{b}) - (\vec{a} \cdot \vec{c})\vec{b}$ . In the third line the gauge condition  $\vec{k} \cdot \vec{A}(\vec{k}, t) = \vec{0}$  was used to eliminate the first term.

To proceed we to put in the table that the polarization vectors satisfy (Lect. Notes, page 109):

$$\begin{aligned}\epsilon(\vec{k}, \lambda) \cdot \epsilon(\vec{k}, \lambda') &= \delta_{\lambda\lambda'} \\ \epsilon(\vec{k}, \lambda) \cdot \epsilon(-\vec{k}, \lambda') &= (-1)^\lambda \delta_{\lambda\lambda'}\end{aligned}$$

This means that apart from the fact that  $\epsilon(\vec{k}, \lambda)$  for  $\lambda = 1, 2$  being orthonormal, that  $\epsilon(-\vec{k}, 1) \uparrow \uparrow \epsilon(\vec{k}, 1)$  and  $\epsilon(-\vec{k}, 1) \uparrow \downarrow \epsilon(\vec{k}, 1)$ .

Now that everything is set up lets evaluate the electric field term:

$$\begin{aligned}\frac{1}{c} \frac{\partial}{\partial t} \vec{A}(\vec{k}, t) \cdot \frac{1}{c} \frac{\partial}{\partial t} \vec{A}(-\vec{k}, t) &= \frac{4\pi\hbar^2}{2E(\vec{k})} \frac{i^2 E^2(\vec{k})}{\hbar^2} \sum_{\lambda=1,2} \left[ -\vec{e}(-\vec{k}, \lambda) \hat{a}(-\vec{k}, \lambda) e^{-iE(\vec{k})t/\hbar} + \vec{e}(\vec{k}, \lambda) \hat{a}(\vec{k}, \lambda)^\dagger e^{iE(\vec{k})t/\hbar} \right] \\ &\quad \sum_{\lambda'=1,2} \left[ -\vec{e}(\vec{k}, \lambda') \hat{a}(\vec{k}, \lambda') e^{-iE(\vec{k})t/\hbar} + \vec{e}(-\vec{k}, \lambda') \hat{a}(-\vec{k}, \lambda')^\dagger e^{iE(\vec{k})t/\hbar} \right] \\ &= -2\pi E(\vec{k}) \sum_{\lambda} (-1)^\lambda \left[ \hat{a}(-\vec{k}, \lambda) \hat{a}(\vec{k}, \lambda) e^{-2iE(\vec{k})t/\hbar} + \hat{a}(\vec{k}, \lambda)^\dagger \hat{a}(-\vec{k}, \lambda)^\dagger e^{-2iE(\vec{k})t/\hbar} \right] \\ &\quad + 2\pi E(\vec{k}) \sum_{\lambda} \left[ \hat{a}(-\vec{k}, \lambda) \hat{a}(-\vec{k}, \lambda)^\dagger + \hat{a}(\vec{k}, \lambda)^\dagger \hat{a}(\vec{k}, \lambda) \right]\end{aligned}$$

Calculating the  $\vec{k}^2 \vec{A}(\vec{k}, t) \cdot \vec{A}(-\vec{k}, t)$  term is the same. The difference is that all terms will appear with a positive sign, because there is no  $i^2$  term and no minus sign due to exponent in the first term of each vector potential. Also the prefactor will be  $\frac{4\pi\hbar^2 c^2}{2E(\vec{k})} |\vec{k}|^2 = 2\pi E(\vec{k})$  which is the same. In other words we will have the same expression but in the first term we will have a plus sign instead and the two time dependent terms (from the electric and the magnetic field) will cancel. The expression for the energy is going to be:

$$H = \frac{1}{8\pi} \int d^3\vec{k} 4\pi E(\vec{k}) \sum_{\lambda} \left[ \hat{a}(-\vec{k}, \lambda) \hat{a}(-\vec{k}, \lambda)^\dagger + \hat{a}(\vec{k}, \lambda)^\dagger \hat{a}(\vec{k}, \lambda) \right]$$

where in the second line we remembered the commutation relation which implies that  $\hat{a}(-\vec{k}, \lambda) \hat{a}(-\vec{k}, \lambda)^\dagger = 1 + \hat{a}(-\vec{k}, \lambda)^\dagger \hat{a}(-\vec{k}, \lambda)$ . The constant will give the vacuum energy which in this calculation diverges (there is a really really long story about how to deal with this). In the second term we perform the  $\vec{k} \rightarrow -\vec{k}$  transformation in the summation and exploit the fact that  $E(-\vec{k}) = E(\vec{k})$ . The energy finally reads:

$$H = \int d^3\vec{k} \sum_{\lambda} E(\vec{k}) \hat{n}(\vec{k}, \lambda)$$

where  $\hat{n}(\vec{k}, \lambda) = \hat{a}(\vec{k}, \lambda)^\dagger \hat{a}(\vec{k}, \lambda)$  is the number operator. Intuitively the energy is just a sum of the energies  $E(\vec{k})$  of all the occupied modes.

Using the same notation we will now evaluate the momentum operator:

$$P = -\frac{1}{4\pi c^2} \int d^3\vec{r} \frac{\partial \vec{A}}{\partial t} \times (\nabla \times \vec{A})$$

As before we will use equation 2 to perform the spacial integration:

$$\begin{aligned}
P &= -\frac{1}{4\pi c^2} \int d^3\vec{k} \frac{\partial \vec{A}(\vec{k}, t)}{\partial t} \times \left( i\vec{k} \times \vec{A}(-\vec{k}, t) \right) \\
&= -\frac{i}{4\pi c^2} \int d^3\vec{k} \left[ \left( \frac{\partial \vec{A}(\vec{k}, t)}{\partial t} \cdot \vec{A}(-\vec{k}, t) \right) \vec{k} - \frac{\partial}{\partial t} \left( \vec{k} \cdot \vec{A}(\vec{k}, t) \right) \vec{A}(-\vec{k}, t) \right]
\end{aligned}$$

The second term is zero due to choice of gauge. The first term, after doing the same kind of algebra as we did for the energy will become:

$$\begin{aligned}
\frac{1}{c^2} \frac{\partial}{\partial t} \vec{A}(\vec{k}, t) \cdot \vec{A}(-\vec{k}, t) &= \frac{4\pi\hbar^2}{2E(\vec{k})} \frac{iE(\vec{k})}{\hbar} \sum_{\lambda=1,2} \left[ -\bar{c}(-\vec{k}, \lambda) \hat{a}(-\vec{k}, \lambda) e^{-iE(\vec{k})t/\hbar} + \bar{c}(\vec{k}, \lambda) \hat{a}(\vec{k}, \lambda)^\dagger e^{iE(\vec{k})t/\hbar} \right] \\
&\quad \sum_{\lambda'=1,2} \left[ \bar{c}(\vec{k}, \lambda') \hat{a}(\vec{k}, \lambda') e^{-iE(\vec{k})t/\hbar} + \bar{c}(-\vec{k}, \lambda') \hat{a}(-\vec{k}, \lambda')^\dagger e^{iE(\vec{k})t/\hbar} \right] \\
&= 2\pi i\hbar \sum_{\lambda} (-1)^\lambda \left[ -\hat{a}(-\vec{k}, \lambda) \hat{a}(\vec{k}, \lambda) e^{-2iE(\vec{k})t/\hbar} + \hat{a}(\vec{k}, \lambda)^\dagger \hat{a}(-\vec{k}, \lambda)^\dagger e^{-2iE(\vec{k})t/\hbar} \right] \\
&\quad + 2\pi i\hbar \sum_{\lambda} \left[ -\hat{a}(-\vec{k}, \lambda) \hat{a}(-\vec{k}, \lambda)^\dagger + \hat{a}(\vec{k}, \lambda)^\dagger \hat{a}(\vec{k}, \lambda) \right]
\end{aligned}$$

All of this is multiplied by a  $\vec{k}$  in the integral. The term  $\hat{a}(-\vec{k}, \lambda) \hat{a}(\vec{k}, \lambda) \vec{k}$  changes sign under a  $\vec{k} \rightarrow -\vec{k}$  because the two operators commute. Therefore in the integral this term will contribute zero. Same for the  $\hat{a}^\dagger \hat{a}^\dagger$  term. For the second line we also perform a  $\vec{k} \rightarrow -\vec{k}$  for the first term to get:

$$\begin{aligned}
P &= -\frac{i}{4\pi} 2\pi i c\hbar \int d^3\vec{k} \sum_{\lambda} \left[ \vec{k} \hat{a}(\vec{k}, \lambda) \hat{a}(\vec{k}, \lambda)^\dagger + \vec{k} \hat{a}(\vec{k}, \lambda)^\dagger \hat{a}(\vec{k}, \lambda) \right] \\
&= \frac{1}{2} c\hbar \int d^3\vec{k} \sum_{\lambda} \left[ \vec{k} + 2\vec{k} \hat{a}(\vec{k}, \lambda)^\dagger \hat{a}(\vec{k}, \lambda) \right] \\
&= \int d^3\vec{k} \sum_{\lambda} \hbar c \vec{k} \hat{n}(\vec{k}, \lambda)
\end{aligned}$$

In the second line we used the commutation relation and in the third line the fact that  $\int \vec{k} d^3\vec{k} = 0$ . Again the total momentum is the sum of the momenta  $\hbar\vec{k}$  of the occupied modes.

## 2.

By commuting the just terms that correspond to the same mode one can easily show that:

$$\begin{aligned}
[A_i(\vec{r}, t), A_j(\vec{r}', t)] &= 0 \\
[E_i(\vec{r}, t), E_j(\vec{r}', t)] &= 0
\end{aligned}$$

Both the electric field and the vector potential at the same time and at different points commute. For the magnetic field we don't need to evaluate anything: the magnetic field is just the spacial derivative of the vector

potential. If we realise that the derivative is nothing but the limit of the difference of the vector potential between neighboring points and at the same time, we can argue that the magnetic field is the limit of a linear combination of vector potentials at different points. Since the vector potential between any points commutes, so does the magnetic field:

$$[B_i(\vec{r}, t), B_j(\vec{r}', t)] = 0$$

For the commutator of the electric field and the vector potential we need to do a bit more work. The electric field is give by:

$$\vec{E}(\vec{k}, t) = -\frac{1}{c} \frac{\partial \vec{A}}{\partial t} = \frac{i}{\hbar c} \int \frac{d^3 \vec{k} \sqrt{4\pi \hbar c}}{\sqrt{(2\pi)^3 2E(\vec{k})}} E(\vec{k}) \sum_{\lambda=1,2} \vec{\epsilon}(\vec{k}, \lambda) \left[ \hat{a}(\vec{k}, \lambda) e^{i\vec{k} \cdot \vec{r} - iE(\vec{k})t/\hbar} - \hat{a}(\vec{k}, \lambda)^\dagger e^{-i\vec{k} \cdot \vec{r} + iE(\vec{k})t/\hbar} \right] \quad (3)$$

Using this expression along with the commutation relations will give:

$$[E_i(\vec{r}, t), A_j(\vec{r}', t)] = \frac{i}{\hbar c} \frac{4\pi \hbar^2 c^2}{(2\pi)^{3/2}} \int d^3 \vec{k} \sum_{\lambda=1,2} \epsilon_i(\vec{k}, \lambda) \epsilon_j(\vec{k}, \lambda) \left[ [\hat{a}(\vec{k}, \lambda), \hat{a}(\vec{k}, \lambda)^\dagger] e^{i\vec{k} \cdot (\vec{r} - \vec{r}')} - [\hat{a}(\vec{k}, \lambda)^\dagger, \hat{a}(\vec{k}, \lambda)] e^{-i\vec{k} \cdot (\vec{r} - \vec{r}')} \right]$$

where  $\epsilon_i = \hat{e}_i \cdot \epsilon$  with  $i = 1, 2, 3$  is the  $i^{\text{th}}$  component of  $\vec{\epsilon}$ . First we will perform the  $\lambda$  summation. Lets fix some  $\vec{k}$  and with no loss of generality choose a system of coordinates such that  $\epsilon_1 \rightarrow (1, 0, 0)$  and  $\epsilon_2 \rightarrow (0, 1, 0)$  (the  $z$  direction is the the direction of  $\vec{k}$ ). Also let  $\vec{e}_i$  and  $\vec{e}_j$  be two arbitrary vectors represented by column vectors. Then:

$$\begin{aligned} \sum_{\lambda=1,2} \epsilon_i(\vec{k}, \lambda) \epsilon_j(\vec{k}, \lambda) &= \vec{e}_i^{\dagger r} \cdot \left[ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \cdot (1 \ 0 \ 0) + \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \cdot (0 \ 1 \ 0) \right] \cdot \vec{e}_j \\ &= \vec{e}_i^{\dagger r} \cdot \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \cdot \vec{e}_j = \vec{e}_i \cdot \vec{e}_j - (\vec{e}_i \cdot \hat{k})(\vec{e}_j \cdot \hat{k}) \end{aligned}$$

where  $\hat{k}$  is the direction of  $\vec{k}$ . If  $\vec{e}_i$  and  $\vec{e}_j$  are orthonormal we can write:

$$\sum_{\lambda=1,2} \epsilon_i(\vec{k}, \lambda) \epsilon_j(\vec{k}, \lambda) = \delta_{ij} - \frac{k_i k_j}{k^2}$$

Substituting gives:

$$\begin{aligned} [E_i(\vec{r}, t), A_j(\vec{r}', t)] &= i \frac{2\pi \hbar c}{(2\pi)^3} \int d^3 \vec{k} \left( \delta_{ij} - \frac{k_i k_j}{k^2} \right) \left[ e^{i\vec{k} \cdot (\vec{r} - \vec{r}')} + e^{-i\vec{k} \cdot (\vec{r} - \vec{r}')} \right] \\ &= i 4\pi \hbar c \int \frac{d^3 \vec{k}}{(2\pi)^3} \left( \delta_{ij} - \frac{k_i k_j}{k^2} \right) e^{i\vec{k} \cdot (\vec{r} - \vec{r}')} \end{aligned}$$

In the last line we just set  $\vec{k} \rightarrow -\vec{k}$  in the second term. This integral is called transverse commutator. The magnetic field is obtained by  $B_m = \varepsilon_{ajm} \partial'_a A_j$  and the electric-magnetic field commutator is given by:

$$[E_i(\vec{r}, t), B_m(\vec{r}', t)] = i 4\pi \hbar c \int \frac{d^3 \vec{k}}{(2\pi)^3} \left( \delta_{ij} - \frac{k_i k_j}{k^2} \right) (-i) e^{i\vec{k} \cdot (\vec{r} - \vec{r}')} \varepsilon_{ajm} k_a$$

but we get  $\varepsilon_{ajm}k_a k_j = (\vec{k} \times \vec{k})_m = 0$ . The commutator is then:

$$[E_i(\vec{r}, t), B_m(\vec{r}', t)] = i4\pi\hbar c \int \frac{d^3\vec{k}}{(2\pi)^3} i \left[ -e^{i\vec{k}\cdot(\vec{r}-\vec{r}')} + e^{-i\vec{k}\cdot(\vec{r}-\vec{r}')} \right] \varepsilon_{aim}k_a = 4\pi\hbar c \varepsilon_{aim} i \partial'_a \delta(\vec{r} - \vec{r}')$$

### 3

First we will need the commutators:

$$\begin{aligned} \left[ \hat{a}(\vec{k}', \lambda'), \hat{a}(\vec{k}, \lambda)^\dagger \hat{a}(\vec{k}, \lambda) \right] &= \hat{a}(\vec{k}, \lambda) \delta_{\lambda\lambda'} \delta(\vec{k} - \vec{k}') \\ \left[ \hat{a}(\vec{k}', \lambda')^\dagger, \hat{a}(\vec{k}, \lambda)^\dagger \hat{a}(\vec{k}, \lambda) \right] &= -\hat{a}(\vec{k}, \lambda)^\dagger \delta_{\lambda\lambda'} \delta(\vec{k} - \vec{k}') \end{aligned}$$

Integrating over  $\vec{k}$  and summing over  $\lambda$  gives:

$$\begin{aligned} \left[ \hat{a}(\vec{k}, \lambda) e^{-iE(\vec{k})t/\hbar}, \hat{N} \right] &= \hat{a}(\vec{k}, \lambda) e^{-iE(\vec{k})t/\hbar} = \frac{i}{E(\vec{k})} \partial_t \hat{a}(\vec{k}, \lambda) e^{-iE(\vec{k})t/\hbar} \\ \left[ \hat{a}(\vec{k}, \lambda)^\dagger e^{iE(\vec{k})t/\hbar}, \hat{N} \right] &= -\hat{a}(\vec{k}, \lambda)^\dagger e^{iE(\vec{k})t/\hbar} = \frac{i}{E(\vec{k})} \partial_t \hat{a}(\vec{k}, \lambda)^\dagger e^{iE(\vec{k})t/\hbar} \end{aligned}$$

In either case taking the commutator with the number operator is like taking the time derivative and dividing by the energy of the mode. If a commutator vanishes it vanishes identically for each mode separately. Clearly this cannot happen by just taking a time derivative so neither the electric or the magnetic field commute with the number operator. The physical interpretation for this is that the number of photons is not conserved when the electromagnetic field is coupled to matter.

### 4.

To verify those equations it suffices to consider one mode. In the same way that we took the commutator with the number operator at part (3) we can take the commutator with the total energy. The only difference is that we multiply by  $E(\vec{k})$  before integrating over  $\vec{k}$  and summing over  $\lambda$ . The result is:

$$\begin{aligned} \left[ \hat{a}(\vec{k}, \lambda) e^{-iE(\vec{k})t/\hbar}, \hat{H} \right] &= E(\vec{k}) \hat{a}(\vec{k}, \lambda) e^{-iE(\vec{k})t/\hbar} = i \partial_t \hat{a}(\vec{k}, \lambda) e^{-iE(\vec{k})t/\hbar} \\ \left[ \hat{a}(\vec{k}, \lambda)^\dagger e^{iE(\vec{k})t/\hbar}, \hat{H} \right] &= -E(\vec{k}) \hat{a}(\vec{k}, \lambda)^\dagger e^{iE(\vec{k})t/\hbar} = i \partial_t \hat{a}(\vec{k}, \lambda)^\dagger e^{iE(\vec{k})t/\hbar} \end{aligned}$$

In both cases, commuting with the Hamiltonian is like taking the derivative. Same applies to any operator that can be expressed as a linear combination of  $a$  and  $a^\dagger$  like the electric field, the magnetic field, the vector potential etc. There are two Maxwell equations that involve the time derivatives:

$$\begin{aligned} \vec{\nabla} \times \vec{E} &= -\frac{1}{c} \frac{\partial \vec{B}}{\partial t} \Rightarrow \vec{\nabla} \times \vec{E} = -\frac{1}{c} \vec{\nabla} \times \frac{\partial \vec{A}}{\partial t} \\ \vec{\nabla} \times \vec{B} &= \frac{1}{c} \frac{\partial \vec{E}}{\partial t} \Rightarrow \frac{1}{c^2} \frac{\partial^2 \vec{A}}{\partial t^2} = -\vec{\nabla} (\vec{\nabla} \cdot \vec{A}) + \vec{\nabla}^2 \vec{A} \end{aligned}$$

Those correspond to the cases that there is no source current.

The first one, along with  $\vec{B} = \vec{\nabla} \times \vec{A}$  implies that  $\vec{E} = -\frac{1}{c} \frac{\partial}{\partial t} \vec{A} - \vec{\nabla} \phi$  but in the absence of charges the second term vanishes. In momentum space the second equation will become:

$$\frac{\partial^2 \vec{A}(\vec{k}, t)}{\partial t^2} = c^2 \vec{k}^2 \vec{A}(\vec{k}, t)$$

which is automatically satisfied because the time dependence of each mode is  $e^{\pm iE(\vec{k})t/\hbar}$  with  $E(\vec{k}) = \hbar c |\vec{k}|$ .

## 5.

This is a tricky question. First  $\vec{\nabla} \cdot \vec{B}$  is identically zero. Also  $\vec{\nabla} \cdot \vec{E} = -\frac{1}{c} \frac{\partial}{\partial t} \vec{\nabla} \cdot \vec{A} = 0$  in the given gauge. One can see that formally in Fourier space:  $\vec{k} \cdot \vec{E}(\vec{k}, t) = 0$  because the polarization vectors are perpendicular to  $\vec{k}$ . The results mean that there are no sources of electric or magnetic field (free space). Therefore all commutators are identically zero.

## 6.

The normalized wave functions are:

$$|\vec{p}, R|L\rangle = a^\dagger(\vec{k}, R|L) |0\rangle \quad (4)$$

Consulting the Lect.. Notes page 118, we get that:

$$\begin{aligned} a^\dagger(\vec{k}, 1) &= \frac{a^\dagger(\vec{k}, R) + a^\dagger(\vec{k}, L)}{\sqrt{2}} \\ a^\dagger(\vec{k}, 2) &= \frac{a^\dagger(\vec{k}, R) - a^\dagger(\vec{k}, L)}{i\sqrt{2}} \end{aligned}$$

It is straightforward to show that in the  $R, L$  basis the operators satisfy the same commutation relations. Using this fact we can show that:

$$\langle \vec{p}, \mu | \vec{p}', \mu' \rangle = \langle 0 | a(\vec{p}, \mu) a^\dagger(\vec{p}', \mu') | 0 \rangle = \langle 0 | \delta_{\mu\mu'} \delta(\vec{p} - \vec{p}') + a^\dagger(\vec{p}', \mu') a(\vec{p}, \mu) | 0 \rangle = \delta_{\mu\mu'} \delta(\vec{p} - \vec{p}')$$

where  $\mu = R, L$ . Therefore the above states are correctly normalized.

Now we will obtain expressions of the Hamiltonian, momentum and angular momentum using the  $L, R$  polarization vectors.

The number operators will become:

$$\begin{aligned} \hat{n}(\vec{k}, 1) &= \frac{a^\dagger(\vec{k}, R) + a^\dagger(\vec{k}, L)}{\sqrt{2}} \frac{a(\vec{k}, R) + a(\vec{k}, L)}{\sqrt{2}} = \frac{\hat{n}(\vec{k}, R) + \hat{n}(\vec{k}, L) + a^\dagger(\vec{k}, L) a(\vec{k}, R) + a^\dagger(\vec{k}, R) a(\vec{k}, L)}{2} \\ \hat{n}(\vec{k}, 2) &= \frac{a^\dagger(\vec{k}, R) - a^\dagger(\vec{k}, L)}{i\sqrt{2}} \frac{a(\vec{k}, R) - a(\vec{k}, L)}{-i\sqrt{2}} = \frac{\hat{n}(\vec{k}, R) + \hat{n}(\vec{k}, L) - a^\dagger(\vec{k}, L) a(\vec{k}, R) - a^\dagger(\vec{k}, R) a(\vec{k}, L)}{2} \end{aligned}$$

From this we get:

$$\sum_{\lambda=1,2} \hat{n}(\vec{k}, \lambda) = \sum_{\mu=L,R} \hat{n}(\vec{k}, \mu)$$

Using this we can quickly get the same expressions for the momentum and the Hamiltonian, only this time the polarization summation is over  $R$  and  $L$ :

$$H = \int d^3\vec{k} \sum_{\mu=R,L} E(\vec{k}) \hat{n}(\vec{k}, \mu)$$

$$\vec{P} = \int d^3\vec{k} \sum_{\mu=R,L} \hbar\vec{k} \hat{n}(\vec{k}, \mu)$$

The algebra for obtaining the angular momentum is already in the Lect.. Notes in pages 118-119:

$$\vec{L} = \int d^3\vec{k} \hbar\vec{k} \left( \hat{n}(\vec{k}, R) - \hat{n}(\vec{k}, L) \right)$$

All the operators  $H$ ,  $\vec{P}$  and  $\vec{L}$  have the same general form:

$$Q = \int d^3\vec{k} \sum_{\mu=R,L} G(\vec{k}, \mu) \hat{n}(\vec{k}, \mu)$$

Before applying all those operators in the states 4 we need one more little results:

$$\hat{n}(\vec{p}', \mu') a^\dagger(\vec{p}, \mu) |0\rangle = \hat{a}(\vec{p}', \mu')^\dagger \delta_{\mu\mu'} \delta(\vec{p} - \vec{p}') |0\rangle + \hat{a}(\vec{p}', \mu')^\dagger a^\dagger(\vec{p}, \mu) \hat{a}(\vec{p}', \mu') |0\rangle = \delta_{\mu\mu'} \delta(\vec{p} - \vec{p}') \hat{a}(\vec{p}, \mu)^\dagger |0\rangle$$

Applying the general  $Q$  operator will give:

$$Q |\vec{p}, \mu\rangle = \int d^3\vec{p}' \sum_{\mu'=R,L} G(\vec{p}', \mu') \delta_{\mu\mu'} \delta(\vec{p} - \vec{p}') \hat{a}(\vec{p}, \mu)^\dagger |0\rangle = G(\vec{p}, \mu) |\vec{p}, \mu\rangle$$

Therefore applying  $Q$  on the single photon state just picks the corresponding term. Therefore the states 4 have energy  $E(\vec{k})$ , momentum  $\hbar\vec{k}$  and angular momentum  $+\hbar$  in the  $R$  polarization and  $-\hbar$  in the  $L$  polarization.

## 2 Scattering of Light

### 1.

The incoming state has  $N$  photons all with the same momentum  $\vec{p}$  and polarization  $\nu$ . The outgoing photon state has one photon removed from the  $(\vec{p}, \nu)$  mode and one single photon in the mode  $(\vec{p}', \nu')$ . The correctly normalized photon states can be constructed with the creation operators for photons.

$$|I_{photon}\rangle = |N, 0\rangle = \sqrt{\frac{(2\pi)^3}{V}} \frac{a^{\dagger N}(\vec{p}, \nu)}{\sqrt{N!}} |0\rangle$$

$$|F_{photon}\rangle = |N-1, 1\rangle = \sqrt{\frac{(2\pi)^3}{V}} \frac{a^{\dagger N-1}(\vec{p}, \nu) a^\dagger(\vec{p}', \nu')}{\sqrt{(N-1)!}} |0\rangle$$

The first slot corresponds to the  $(\vec{p}, \nu)$  mode and the second to the  $(\vec{p}', \nu')$  mode. This way we avoid the rather confusing  $N(\vec{p}', \nu')$  and  $N(\vec{p}, \nu)$  notation.

## 2.

In case the volume is finite the  $k$ -integration is replaced by a summation using the rule:

$$\int d^3\vec{k} \rightarrow \frac{(2\pi)^3}{V} \sum_k$$

Also the creation/annihilation operators are renormalized by a factor  $\sqrt{\frac{V}{(2\pi)^3}}$ . The vector potential reads:

$$\vec{A}(\vec{r}, t) = \sum_{\vec{k}} \frac{\sqrt{4\pi\hbar c}}{\sqrt{V2E(\vec{k})}} \sum_{\lambda=1,2} \vec{\epsilon}(\vec{k}, \lambda) \left[ \hat{a}(\vec{k}, \lambda) e^{i\vec{k}\cdot\vec{r} - iE(\vec{k})t/\hbar} + \hat{a}(\vec{k}, \lambda)^\dagger e^{-i\vec{k}\cdot\vec{r} + iE(\vec{k})t/\hbar} \right] \quad (5)$$

The matrix element we need is

$$\langle F_{photon} | \vec{A}^2(\vec{r}) | I_{photon} \rangle$$

For the element to be non zero, in the initial state exactly one  $a^\dagger(\vec{p}', \nu')$  and one  $a(\vec{p}, \nu)$  should be applied to give the correct final state. Those can appear in any order and we have two possibilities:  $a^\dagger(\vec{p}', \nu')a(\vec{p}, \nu)$  or  $a(\vec{p}, \nu)a^\dagger(\vec{p}', \nu') = \delta_{\nu\nu'}\delta(\vec{p}-\vec{p}') + a^\dagger(\vec{p}', \nu')a(\vec{p}, \nu)$ . Because the initial and final states have different occupations the  $\delta_{\nu\nu'}\delta(\vec{p}-\vec{p}')$  term between them will vanish, so both probabilities will give the same result. This gives us a factor of 2. If we also put all the other factors that accompany the creation/annihilation operators in the expression of the vector potential we will get:

$$2 \frac{4\pi\hbar^2 c^2}{V\sqrt{4E(\vec{p})E(\vec{p}')}} \vec{\epsilon}(\vec{p}', \nu') \cdot \vec{\epsilon}(\vec{p}, \nu) e^{i\vec{r}\cdot(\vec{p}-\vec{p}')} \langle F_{photon} | a^\dagger(\vec{p}', \nu')a(\vec{p}, \nu) | I_{photon} \rangle$$

To proceed we will apply the operators (conjugated) to the left, that is the Final photon state:

$$a(\vec{p}, \nu)^\dagger a(\vec{p}', \nu') |N-1, 1\rangle = \sqrt{N}\sqrt{1} |N, 0\rangle$$

We are getting for the matrix element the expression:

$$\frac{4\pi\hbar c^2 \sqrt{N} \vec{\epsilon}(\vec{p}', \nu') \cdot \vec{\epsilon}(\vec{p}, \nu) e^{i\vec{r}\cdot(\vec{p}-\vec{p}')}}{V\sqrt{\omega(\vec{p})\omega(\vec{p}')}}$$

## 3.

The density operator affects the degrees of freedom of the atom but not the the electromagnetic field. Therefore we can perform the factorization:

$$\begin{aligned} M = \langle n; N-1, 1 | H_{int}^{(2)} | 0; N, 0 \rangle &= \frac{r_0}{2} \langle n | \int d^3\vec{r} \rho(\vec{r}) \langle F_{photon} | \vec{A}^2(\vec{r}) | I_{photon} \rangle | n = 0 \rangle \\ &= \frac{2r_0\pi\hbar c^2 \sqrt{N} \vec{\epsilon}(\vec{p}', \nu') \cdot \vec{\epsilon}(\vec{p}, \nu)}{V\sqrt{\omega(\vec{p})\omega(\vec{p}')}} \langle n | \int d^3\vec{r} \rho(\vec{r}) e^{i\vec{r}\cdot(\vec{p}-\vec{p}')} | n = 0 \rangle \end{aligned}$$

The integral gives nothing by the Fourier transform of the density  $\rho(\vec{p}-\vec{p}')$ .

#### 4.

The differential cross section is given by the transition rate (Fermi's Golden Rule) divided by the incoming photon flux:

$$\frac{d\sigma_{0 \rightarrow n}}{d\Omega} = \frac{V}{cN} \frac{2\pi}{\hbar} |M|^2 D(E')$$

where  $D(E')$  is the density of states of the outgoing photon with energy  $E' = E(\vec{p}') = E_n - E_0 + E(\vec{p})$ . The constraints are the the photon should move in a infinitesimal solid angle  $d\Omega$ .

The phonon density of states is given by:

$$\frac{V}{\hbar^3} d^3\vec{p} = \frac{V}{\hbar^3} p^2 dp d\Omega = \frac{V}{\hbar^3 c^3} E^2 dE d\Omega = D(E) dE d\Omega$$

(NOTE:  $h$  is not  $\hbar$ ).

Then the differential cross section becomes:

$$\begin{aligned} \frac{d\sigma_{0 \rightarrow n}}{d\Omega} &= \frac{V}{cN} \frac{2\pi}{\hbar} |M|^2 \frac{V}{\hbar^3 c^3} E'^2 = \\ &= r_0^2 \frac{\omega'}{\omega} |\vec{\epsilon}(\vec{p}', \nu') \cdot \vec{\epsilon}(\vec{p}, \nu)|^2 |\langle n | \rho(\vec{p} - \vec{p}') | 0 \rangle|^2 \end{aligned}$$

#### 5.

If  $\vec{r}'$  is the position of the electron, the density operator is just  $\delta(\vec{r} - \vec{r}')$ . We then have:

$$\langle n | \int d^3\vec{r} \delta(\vec{r} - \vec{r}') e^{i\vec{r} \cdot (\vec{p} - \vec{p}')} | 0 \rangle = \langle n | e^{i\vec{r}' \cdot (\vec{p} - \vec{p}')} | 0 \rangle = \frac{1}{V} \int d^3\vec{r}' e^{-i\vec{q}_n \cdot \vec{r}'} e^{i\vec{r}' \cdot (\vec{p} - \vec{p}')} e^{i\vec{q}_0 \cdot \vec{r}'} = \delta_{\vec{p} + \vec{q}_0, \vec{p}' + \vec{q}_n}$$

The  $\delta$  is a Kronecker delta and not a delta function and it enforces the momentum conservation between the initial and final states. The differential cross section in this case is:

$$\frac{d\sigma_{0 \rightarrow n}}{d\Omega} = r_0^2 \frac{\omega'}{\omega} |\vec{\epsilon}(\vec{p}', \nu') \cdot \vec{\epsilon}(\vec{p}, \nu)|^2 \delta_{\vec{p} + \vec{q}_0, \vec{p}' + \vec{q}_n}$$

with the additional constraint that the energy is conserved in the process:

$$c|\vec{p}| + \frac{\hbar^2 \vec{q}_0^2}{2m} = c|\vec{p}'| + \frac{\hbar^2 \vec{q}_n^2}{2m}$$

### 3 Spontaneous Emission

#### 1.

In this problem we are going to allow both the proton and the electron to move and interact with the electro magnetic field. Therefore there is one more degree of freedom compared to the problem solved in Shankar.

The states of the system are

$$|CM\rangle \oplus |nlm\rangle \oplus |photon\rangle$$

The first part describes the center of mass motion of the neutral atom. The second term describes the internal states of the atom and the third term describes the photon field. The photon part is described by a list of occupation number of the various photon modes (momentum and polarization). The internal states wave function expressed

in the position basis will depend only on the difference  $\vec{r} = \vec{r}_e - \vec{r}_p$ . Finally in the position basis the CM wave function is just a plane wave:  $e^{i\vec{P}\cdot(\vec{r}_e+\vec{r}_p)/\hbar}$  where  $\vec{P}$  is the total momentum. The initial and final states of the system atom-photons are:

$$\begin{aligned} |I\rangle &= \left| \vec{P} = 0 \right\rangle \oplus |211\rangle \oplus |0\rangle \\ |F\rangle &= \left| \vec{P}_f \right\rangle \oplus |100\rangle \oplus \left| \vec{k}\nu \right\rangle \end{aligned}$$

where the polarization of the photon can be  $R$  or  $L$ . Originally the atom will be at rest, but when the photon is emitted there is going to be a recoil and the center of mass momentum will be non zero. Intuitively the photon momentum will be opposite the center of mass momentum of the atom, but we will show that through the algebra. Lets revise the Lect. Notes page 123. Because of the fact that the proton has a different charge than the electron, the Hamiltonian of the system of two particles is:

$$H = \frac{(\vec{p}_p - e\vec{A}(\vec{r}_p, t)/c)^2}{2m_p} + \frac{(\vec{p}_e + e\vec{A}(\vec{r}_e, t)/c)^2}{2m_e}$$

where external potentials have been ignored. From this equation we can get the interaction Hamiltonian by a simple Taylor expansion for small  $\vec{A}$ :

$$\begin{aligned} H_{int} &= -\frac{e}{c} \int d^3\vec{x} \vec{j}(\vec{x}) \cdot \vec{A}(\vec{x}, t) \\ \vec{j}(\vec{k}) &= \frac{1}{2} \left[ \frac{\vec{p}_p}{m_p} e^{-i\vec{k}\cdot\vec{r}_p} + e^{-i\vec{k}\cdot\vec{r}_p} \frac{\vec{p}_p}{m_p} - \frac{\vec{p}_e}{m_e} e^{-i\vec{k}\cdot\vec{r}_e} - e^{-i\vec{k}\cdot\vec{r}_e} \frac{\vec{p}_e}{m_e} \right] \end{aligned}$$

The current is expressed through its Fourier transform. Note that the terms in the current that correspond to the electron come with a different sign due to the opposite charge. Now lets define  $M\vec{R} = m_p\vec{r}_p + m_e\vec{r}_e$  and  $\vec{r} = \vec{r}_e - \vec{r}_p$  with  $M = m_e + m_p$  so that  $\vec{r}_e = \vec{R} + \frac{m_p}{M}\vec{r}$  and  $\vec{r}_p = \vec{R} - \frac{m_e}{M}\vec{r}$ . For the transformation to be canonical we also have to set  $\vec{P} = \vec{p}_e + \vec{p}_p$  and  $\vec{p}_\mu = \frac{\vec{p}_e}{m_e} - \frac{\vec{p}_p}{m_p}$  which means that  $\frac{\vec{p}_e}{m_e} = \frac{\vec{P}}{M} + \frac{\vec{p}}{m_e}$  and  $\frac{\vec{p}_p}{m_p} = \frac{\vec{P}}{M} - \frac{\vec{p}}{m_p}$ . The capital operators commute with all the small operators (the center of mass degrees of freedom are irrelevant to the internal degrees of freedom). We can express everything in terms of the new operator and the current will become:

Replacing the will give the current:

$$\begin{aligned} 2\vec{j}(\vec{k}) &= \left( \frac{\vec{P}}{M} - \frac{\vec{p}}{m_p} \right) e^{-i\vec{k}\cdot\vec{R}} e^{i\vec{k}\cdot\vec{r}m_e/M} + e^{i\vec{k}\cdot\vec{r}m_e/M} e^{-i\vec{k}\cdot\vec{R}} \left( \frac{\vec{P}}{M} - \frac{\vec{p}}{m_p} \right) \\ &\quad - \left( \frac{\vec{P}}{M} + \frac{\vec{p}}{m_e} \right) e^{-i\vec{k}\cdot\vec{R}} e^{-i\vec{k}\cdot\vec{r}_e m_p/M} - e^{-i\vec{k}\cdot\vec{r}_e m_p/M} e^{-i\vec{k}\cdot\vec{R}} \left( \frac{\vec{P}}{M} + \frac{\vec{p}}{m_e} \right) \\ &= \left( \frac{\vec{P}}{M} e^{-i\vec{k}\cdot\vec{R}} + e^{-i\vec{k}\cdot\vec{R}} \frac{\vec{P}}{M} \right) \left( e^{i\vec{k}\cdot\vec{r}m_e/M} - e^{-i\vec{k}\cdot\vec{r}m_p/M} \right) \\ &\quad - e^{-i\vec{k}\cdot\vec{R}} \left( \frac{\vec{p}}{m_p} e^{i\vec{k}\cdot\vec{r}m_e/M} + e^{i\vec{k}\cdot\vec{r}m_e/M} \frac{\vec{p}}{m_p} + \frac{\vec{p}}{m_e} e^{-i\vec{k}\cdot\vec{r}m_p/M} + e^{-i\vec{k}\cdot\vec{r}m_p/M} \frac{\vec{p}}{m_e} \right) \end{aligned}$$

Now we need to evaluate the transition matrix element:

$$\langle F | H_{int} | I \rangle$$

The first step is to evaluate the transition element for the photons  $\langle \vec{k}\nu | \vec{A}(\vec{x}, t) | 0 \rangle$ . We can just follow exactly the algebra in the Lect. Notes to get the result of page 126:

$$\langle 0; \vec{k}\lambda | H_{int} | 0; 0 \rangle = - \sqrt{\frac{2\pi\hbar e^2}{V\omega(\vec{k})}} \langle 0 | \vec{j}(\vec{k}) | n \rangle \cdot \vec{\epsilon}(\vec{k}, \nu)$$

here I use the notation  $|0\rangle$  and  $|n\rangle$  to represent the final and initial state of the atom, including both the internal and the center of mass degrees of freedom. Now we have to evaluate this matrix element. The job is not as hard as it seems. We start by integrating out the CM degrees of freedom:

$$\left\langle \vec{P}_f \left| \frac{\vec{P}}{M} e^{-i\vec{k}\cdot\vec{R}} + e^{-i\vec{k}\cdot\vec{R}} \frac{\vec{P}}{M} \right| \vec{0} \right\rangle = \frac{\vec{P}_f}{M} \langle \vec{P}_f | e^{-i\vec{k}\cdot\vec{R}} | \vec{0} \rangle = \frac{\vec{P}_f}{M} \delta(\vec{P}_f - \hbar\vec{k})$$

This means that after integrating out the CM degrees of freedom we get an effective correct involving only the internal degrees of freedom:

$$\begin{aligned} 2\vec{j}_{eff}(\vec{k})\delta(\vec{P}_f - \hbar\vec{k}) &= \frac{\vec{P}_f}{M} \delta(\vec{P}_f - \hbar\vec{k}) \left( e^{i\vec{k}\cdot\vec{r}_{m_e}/M} - e^{-i\vec{k}\cdot\vec{r}_{m_p}/M} \right) \\ &\quad - \delta(\vec{P}_f - \hbar\vec{k}) \left( \frac{\vec{P}}{m_p} e^{i\vec{k}\cdot\vec{r}_{m_e}/M} + e^{i\vec{k}\cdot\vec{r}_{m_e}/M} \frac{\vec{P}}{m_p} + \frac{\vec{P}}{m_e} e^{-i\vec{k}\cdot\vec{r}_{m_p}/M} + e^{-i\vec{k}\cdot\vec{r}_{m_p}/M} \frac{\vec{P}}{m_e} \right) \end{aligned}$$

This way we got the momentum conservation: the phonon momentum is equal to the atom momentum as we expected. Also inside a box this is not a delta function but a Kronecker delta, so when we square the matrix element it will give a clean Kronecker delta.

There is this identity:

$$e^{-i\vec{a}\cdot\vec{r}} \vec{p} e^{i\vec{a}\cdot\vec{r}} = \vec{p} + \hbar\vec{a}$$

which is similar as the one found in the Lect. Notes page 134. With the help of this we can write the current as:

$$\begin{aligned} 2\vec{j}_{eff}(\vec{k}) &= \frac{\vec{P}_f}{M} \left( e^{i\vec{k}\cdot\vec{r}_{m_e}/M} - e^{-i\vec{k}\cdot\vec{r}_{m_p}/M} \right) \\ &\quad - \left( \frac{1}{m_p} \left[ 2\vec{p} - \frac{m_e\hbar\vec{k}}{M} \right] e^{i\vec{k}\cdot\vec{r}_{m_e}/M} + \frac{1}{m_e} \left[ 2\vec{p} + \frac{m_p\hbar\vec{k}}{M} \right] e^{-i\vec{k}\cdot\vec{r}_{m_p}/M} \right) \\ &= \vec{P}_f \left( \frac{1}{m_p} e^{i\vec{k}\cdot\vec{r}_{m_e}/M} - \frac{1}{m_e} e^{-i\vec{k}\cdot\vec{r}_{m_p}/M} \right) \\ &\quad - 2\vec{p} \left( \frac{1}{m_p} e^{i\vec{k}\cdot\vec{r}_{m_e}/M} + \frac{1}{m_e} e^{-i\vec{k}\cdot\vec{r}_{m_p}/M} \right) \end{aligned}$$

In the second line I used  $\vec{P}_f = \hbar\vec{k}$ . Now it is time for some approximations. Fermi's Golden Rule will give:

$$\Gamma_{211 \rightarrow 100} = \sum_{\nu} \int \frac{V d^3\vec{k}}{(2\pi)^3} \sum_{\vec{P}_f} \frac{4\pi^2 e^2}{V\omega(\vec{k})} \delta(\vec{P}_f - \hbar\vec{k}) \delta\left(E_2 - E_1 - \frac{P_f^2}{2M} - \hbar c |\vec{k}|\right) \left| \langle 100 | \vec{j}_{eff}(\vec{k}) | 211 \rangle \cdot \vec{\epsilon}(\vec{k}, \nu) \right|^2$$

where the energy conservation delta function takes into account both the energy of the recoil and the fact that the photon and atom momentum is the same. The summation extends over all possible final states which are defined by the value of  $\vec{P}_f$  and  $\vec{k}$ . The  $\vec{P}_f$  summation can go first:

$$\Gamma_{211 \rightarrow 100} = \sum_{\nu} \int dk d\Omega \frac{e^2}{2\pi c k} k^2 \delta \left( E_2 - E_1 - \frac{\hbar^2 k^2}{2M} - \hbar c k \right) \left| \left\langle 100 \left| \vec{j}_{eff}(\vec{k}) \right| 211 \right\rangle \cdot \vec{\epsilon}(\vec{k}, \nu) \right|^2$$

The angular integration is not so easy without the matrix element but the  $dk$  integration is trivial. Solving the equation in the delta function will give one positive solution:

$$k_0 = \frac{\Delta E}{\hbar c} \frac{M c^2}{\Delta E} \left( \sqrt{1 + \frac{2\Delta E}{M c^2}} - 1 \right) = \frac{\Delta E}{\hbar c} f \left( \frac{\Delta E}{M c^2} \right) \approx \frac{\Delta E}{\hbar c}$$

where we ignored with not much thought the ratio of the transition energy (few eV) and the relativistic mass of the atom (1GeV).

Then we use the identity  $\delta(f(x)) = \frac{\delta(x-x_0)}{f'(x_0)}$  and  $f(x_0) = 0$  to get:

$$\delta \left( E_2 - E_1 - \frac{\hbar^2 k^2}{2M} - \hbar c k \right) = \delta(k - k_0) \frac{1}{\frac{\hbar^2 k}{M} + \hbar c} = \delta(k - k_0) \frac{1}{\hbar c \sqrt{1 + \frac{2\Delta E}{M c^2}}} \approx \frac{1}{\hbar c} \delta(k - k_0)$$

We also need to simplify the current  $j_{eff}$ . The photon wave length is going to be negligible. More specifically lets compare it with the Bohr radius  $a_0$ :

$$k_0 a_0 = \frac{\Delta E}{\hbar c} a_0 \approx \frac{1}{1726}$$

So the atom is of the order of a thousand times smaller than the wavelength of the photon  $k_0 a_0 \ll 1$ . Multiplying this on both sides by either  $m_e/M$  or  $m_p/M$  only enhances the inequality, so both exponentials can be set to one with great accuracy:

$$\vec{j}_{eff}(\vec{k}) = -\frac{1}{2} \vec{P}_f \left( \frac{1}{m_e} - \frac{1}{m_p} \right) - \frac{\vec{p}}{\mu}$$

Because the transition is between different states the  $\vec{P}_f$  term will not contribute anything. So the transition rate will become:

Then:

$$\Gamma_{211 \rightarrow 100} = \frac{2e^2 \hbar k_0}{\mu^2 c^2 a_0^2} \sum_{\nu} \frac{1}{4\pi} \int d\Omega \left| \left\langle 100 \left| \frac{\vec{p} a_0}{\hbar} \right| 211 \right\rangle \cdot \vec{\epsilon}(\vec{k}, \nu) \right|^2$$

One can proceed by evaluating the integral explicitly using the known analytical expressions for the hydrogen atom wave functions. There is a different way. We can use the Wigner-Eckart theorem (check Sakurai, page 239). Applied here it would give:

$$\langle n'l'm' | p_q | nlm \rangle = \frac{CGC(1q, lm; l'm')}{\sqrt{2j+1}} \Phi(n'l'; nl)$$

where  $CGC(1q, lm; l'm')$  is the Clebsh-Gordan coefficient for adding the angular momenta 1 and  $l$  and  $\Phi(n'l'; nl)$  that does not depend on  $m, m'$  or  $q$ . The vector  $\vec{p}$  is treated as an object of angular momentum one:

$$\begin{aligned} p_0 &= p_z \\ p_1 &= \frac{p_x + ip_y}{\sqrt{2}} \\ p_{-1} &= \frac{p_x - ip_y}{\sqrt{2}} \end{aligned}$$

For the particular quantum numbers the only allowed value for  $q = -1$ . Otherwise the matrix element will be zero. To simplify notation lets define:

$$\vec{a} = \left\langle 100 \left| \frac{\vec{p}a_0}{\hbar} \right| 211 \right\rangle$$

Back to the transition rates: the summation over the polarizations will give (see problem 1 part 2):

$$\Gamma_{211 \rightarrow 100} = \frac{2e^2 \hbar k_0}{\mu^2 c^2 a_0^2} \left[ |\vec{a}|^2 - \frac{1}{4\pi} \int d\Omega |\vec{a} \cdot \hat{k}|^2 \right]$$

Now we need to evaluate something of the form:  $\int d\Omega |\vec{a} \cdot \hat{k}|^2$ . Because we are integrating on the whole sphere we can use spherical coordinates with the z direction along  $\vec{a}$ . Then the integral will become  $|\vec{a}|^2 \int d\Omega \cos^2 \theta = \frac{4\pi}{3} |\vec{a}|^2$ . The vector  $\vec{a} = \langle 100 | \vec{p} | 211 \rangle$  has complex coordinates  $a_x$ ,  $a_y$  and  $a_z$  if we want to perform the rotation to  $a_0$ ,  $a_1$  and  $a_{-1}$ . The norm will be just  $|\vec{a}|^2 = |a_0|^2 + |a_{-1}|^2 + |a_1|^2$ . In this case only  $a_{-1}$  is non zero and the transition rate will become:

$$\Gamma_{211 \rightarrow 100} = \frac{4e^2 \hbar k_0}{3\mu^2 c^2 a_0^2} \left| \left\langle 100 \left| \frac{a_0(\hat{p}_x - i\hat{p}_y)}{\hbar} \right| 211 \right\rangle \right|^2$$

where the number in the brackets is dimensionless.

If we replace  $e^2 = \alpha \hbar c$ ,  $a_0 = \frac{\hbar}{\mu c \alpha}$  and  $E_n = -\frac{\mu c^2 \alpha^2}{2n^2}$ , where  $\alpha = 1/137$  is the fine structure constant, the prefactor can be written as:

$$\frac{4e^2 \hbar k_0}{3\mu^2 c^2 a_0^2} = \frac{4}{3} \frac{\alpha^3 \Delta E}{\hbar} = \frac{\alpha^5 \mu c^2}{2\hbar}$$

which fortunately has exactly the right units.

If we want to maintain the term

$$x = \frac{\Delta E}{Mc^2} = \frac{3}{4} \frac{\mu}{M} \alpha^2$$

all we need to do is multiply the above with:

$$g(x) = \frac{1}{x} (\sqrt{1+x} - 1) \frac{1}{\sqrt{1+x}} = \frac{2}{1+x+\sqrt{1+x}} = 1 - \frac{3}{2}x + \frac{5}{2}x^2 - \frac{35}{8}x^3 + O(x^4)$$

In this case we would get:

$$\Gamma_{211 \rightarrow 100} = \frac{\alpha^5 \mu c^2}{2\hbar} \frac{2}{1+x+\sqrt{1+x}} \left| \left\langle 211 \left| \frac{a_0(\hat{p}_x + i\hat{p}_y)}{\hbar} \right| 100 \right\rangle \right|^2$$

Notice that I swapped the wave function in the integral and I took the hermitian conjugate of the operator.

Now we will try to evaluate the remaining matrix element. Since the only characteristic scale is the Bohr radius and everything else is dimensionless for this calculation we can set  $a_0 = \hbar = 1$ . The relevant wave functions are:

$$\begin{aligned} \psi_{100} &= \frac{1}{\sqrt{\pi}} e^{-r} \\ \psi_{211} &= \frac{1}{\sqrt{64\pi}} r e^{-r/2} = \frac{1}{2\sqrt{6}} r e^{-r/2} Y_1^1 \end{aligned}$$

Applying the momentum operator on the ground state is easy:

$$\begin{aligned}
 p_i \psi_{100} &= -i \frac{e^{-r}}{\sqrt{\pi}} \frac{x_i}{r} \\
 (\hat{p}_x + i\hat{p}_y) \psi_{100} &= i \frac{e^{-r}}{\sqrt{\pi}} \left( \frac{x + iy}{r} \right) = i \frac{e^{-r}}{\sqrt{\pi}} \sin \theta e^{i\phi} = i 2 \sqrt{\frac{2}{3}} e^{-r} Y_1^1
 \end{aligned}$$

The angular integration is trivial it is just  $|Y_1^1| = 1$ . The radial integration will be:

$$\left\langle 211 \left| \frac{a_0(\hat{p}_x + i\hat{p}_y)}{\hbar} \right| 100 \right\rangle = \int_0^\infty dr r^2 i 2 \sqrt{\frac{2}{3}} e^{-r} \frac{1}{2\sqrt{6}} r e^{-r/2} = \frac{i}{3} \int_0^\infty r^3 e^{-3r/2} dr = i \frac{32}{81}$$

And now the final result:

$$\Gamma_{211 \rightarrow 100} = \frac{2^9 \alpha^5 \mu c^2}{3^8 \hbar} \frac{2}{1 + x + \sqrt{1 + x}} \approx 2 \times 10^8 s^{-1}$$

where  $x = \frac{3}{4} \frac{\mu}{M} \alpha^2 \approx 3 \times 10^{-8}$ . The life time is the inverse of the transition rate and is about  $5ns$  and this is the correct order of magnitude for transitions.

## 2 .

Anchoring the proton means taking its mass to infinity so that  $M \rightarrow \infty$  and  $\mu \rightarrow m_e$ . You are asked to compare the result that you got in the first part for these two cases:

$$\frac{\Gamma_{anchored}}{\Gamma_{free}} = \frac{m_e}{\mu} \frac{1 + x + \sqrt{1 + x}}{2} \approx \left( 1 + \frac{m_e}{m_p} \right) \left( 1 + \frac{3}{2} \frac{\Delta E}{Mc^2} \right)$$

The anchored proton has a larger transition rate. The percentage difference is approximately  $100/1836.0\% = 0.05\%$  which is negligible.

## 3.

If the photon leaves from the  $\hat{z}$ -axis its polarization vector will be on the  $\hat{x} - \hat{y}$  plane. Also  $\vec{p} \cdot \vec{\epsilon}(\vec{k}, \nu) = \frac{p_x - ip_y}{\sqrt{2}}$  and we know beforehand that the outgoing photon is going to be right handed and it will have angular momentum  $+\hbar$ . We see that angular momentum is conserved because  $+\hbar$  was also the original angular momentum of the atom.

If the proton is anchored the total momentum is not conserved, the photon momentum is given by whatever holds the proton in place. If the proton is free the total momentum is conserved as the delta function  $\delta(\vec{P}_f - \hbar\vec{k})$  that we derived implies.