

## 1 Counting States of Particles with Different Statistics

### (a) 1 Configuration

One configuration with each fermion occupying a different state.

### (b) 10 Configurations

There is one configuration with all bosons in different states, 6 configurations with two bosons in one state and the third in another, and 3 configurations with all bosons at the same state. In total 10 configurations.

### (c) 9 Configurations

The boson can be in any of the 3 states (3 configurations) and the fermions can fill the states in 3 ways leaving one empty. There bosons do not put any constraint in the configuration of the fermions and in total there are  $3 \times 3 = 9$  configurations.

### (d) 18 Configurations

The fermion can be in any of the 3 states. For the bosons there are 3 configurations with both of them in different states and 3 configurations with both of them in the same state, so there are 6 boson configurations. Since the fermions do not limit the boson occupancy there are in total  $3 \times 6 = 18$  configurations.

### (e) 27 Configurations

If the fermions are distinguishable, Pauli's principle does not apply to them and it does not matter if they are fermions, so each of them can be in any state. There are 27 configurations.

### (f) 27 Configurations

Similarly as in (e) there are 27 configurations.

## 2 Exchange Interaction

### (1)

The matrix element of the Hamiltonian between any two normalized wave functions  $\psi(x) = \langle x|\psi\rangle$  and  $\phi(x) = \langle x|\phi\rangle$  is defined by:

$$\langle \psi|H^{(1)}|\phi\rangle = \int_{-\infty}^{\infty} \psi^*(x) \left( -\frac{\hbar^2}{2M} \frac{d^2}{dx^2} + \frac{U_0}{a^4} (x^2 - a^2)^2 \right) \phi(x) dx$$

The algebra will be already tedious enough so it is worth to be a bit smarter: First we note that  $\langle R|H^{(1)}|L\rangle = \langle L|H^{(1)}|R\rangle^*$  where the complex has been removed because all wave functions are real. So the off diagonal elements are the same. Also the Hamiltonian is symmetric

under  $x \rightarrow -x$ . This operation transforms the  $L$  states to  $R$  states and vice versa. Therefore:  $\langle R|H^{(1)}|R\rangle = \langle L|H^{(1)}|L\rangle$  and the diagonal elements are the same. The Hamiltonian has the form:

$$H_{eff}^{(1)} = \begin{pmatrix} a & b \\ b & a \end{pmatrix}$$

The matrix elements can then be evaluated to:

$$\begin{aligned} a &= \frac{\hbar^2}{8M\xi^2} + U_0 \frac{\xi^2(4a^2 + 3\xi^2)}{a^4} \\ &= \frac{3\hbar^2}{32Ma^2} + \frac{\hbar}{a} \sqrt{\frac{2U_0}{M}} \\ b &= e^{-\frac{a^2}{2\xi^2}} \left( \frac{\hbar^2}{2M} \frac{\xi^2 - a^2}{4\xi^4} + U_0 \frac{a^4 - 2a^2\xi^2 + 3\xi^4}{a^4} \right) \\ &= \frac{3\hbar^2}{32Ma^2} + \frac{\hbar}{2a} \sqrt{\frac{U_0}{2M}} - 3U_0 \end{aligned}$$

where  $\xi = \left( \frac{\hbar^2 a^2}{32MU_0} \right)^{1/4}$

Note that there is a typo in the questions and the wave functions are not normalized to unity: one should generally evaluate the norm of the wave function and not assume that it is one. The normalized wave function is:

$$\psi_{R,L}(x) = \frac{1}{\sqrt{(2\pi)^{1/2}\xi}} e^{-\frac{(x\pm a)^2}{4\xi^2}}$$

If you calculated the matrix elements with the given wave function you need to multiply with  $\sqrt{2\pi}\xi$  to get the correct one. No points will be taken away for this.

### (2)

One can see easily that the effective Hamiltonian can be diagonalized by the linear combination  $|\pm\rangle = \frac{1}{\sqrt{2}}|R\rangle \pm \frac{1}{\sqrt{2}}|L\rangle$  with eigenvalues  $E_{\pm} = a \pm b$  respectively. The overlap is  $l = \langle R|L\rangle = e^{-\frac{a^2}{2\xi^2}}$  (with the wrong normalization you would get a factor of  $\sqrt{2\pi}\xi$  in the denominator). Notice that the overlap is exponentially small. Also the  $|\pm\rangle$  states are normalized to:  $1 \pm l$  and are orthogonal to each other.

### (3)

Including spin there are 4 single particle states:

$$|+, \uparrow\rangle, |+, \downarrow\rangle, |-, \uparrow\rangle, |-, \downarrow\rangle$$

We can guess the number of two particle states by the simple counting argument similar to the one of problem 1: there are 4 single particle states and 2 fermions. The fermions will be distributed in the 4 states so that there is no double occupancy. This can be done in  $\binom{4}{2} = 6$  ways.

To construct the two particle states we start from any product of two single particle states and we antisymmetrize. Some states will vanish during this process (Pauli exclusion principle).

We easily see that if both particles have the same spacial part of the wave function, their total spin must be an antisymmetric singlet, that is we have the states  $|+\rangle_1|+\rangle_2\chi_s$ ,  $|-\rangle_1|-\rangle_2\chi_s$  and  $\frac{1}{\sqrt{2}}(|+\rangle_1|-\rangle_2 + |-\rangle_1|+\rangle_2)\chi_s$  where  $\chi_s = \frac{1}{\sqrt{2}}|\uparrow\rangle_1|\downarrow\rangle_2 - \frac{1}{\sqrt{2}}|\downarrow\rangle_1|\uparrow\rangle_2$ . If the the spacial part of the wave function is symmetric the spin part must be any of the triplet states:  $\frac{1}{\sqrt{2}}(|+\rangle_1|-\rangle_2 - |-\rangle_1|+\rangle_2)\chi_t$  where  $\chi_t$  is one of the spin triplets  $|\uparrow\rangle_1|\uparrow\rangle_2$ ,  $|\downarrow\rangle_1|\downarrow\rangle_2$  or  $\frac{1}{\sqrt{2}}|\uparrow\rangle_1|\downarrow\rangle_2 + \frac{1}{\sqrt{2}}|\downarrow\rangle_1|\uparrow\rangle_2$ . We can express the wave functions formally is we denote by  $|A\rangle$  the single antisymmetric combination of the spacial part and  $|S\rangle$  the three symmetric combinations of the spacial part. Then the only possibilities allowed by the Pauli exclusion principle is:  $|A\rangle\chi_t$  and  $|S\rangle\chi_s$ . We see that there are exactly 6 linearly independent and properly antisymmetrized combinations. Three of them are spin singles and three are spin triplets.

(4)

We want an expression for the matrix element  $V(|x_1 - x_2|)$  ignoring spin again. Lets associate the sign + to R and - to L. So the two states are

$$\psi_\epsilon(x) = \frac{1}{\sqrt{(2\pi)^{1/2}\xi}} e^{-\frac{(x-\epsilon a)^2}{4\xi^2}}$$

where  $\epsilon = \pm 1$ . The matrix elements we need to evaluate have the form:

$$\int dx_1 dx_2 V(|x_1 - x_2|) \psi_{\epsilon_1}(x_1) \psi_{\epsilon_2}(x_2) \psi_{\epsilon_3}(x_1) \psi_{\epsilon_4}(x_2)$$

The first obvious simplification is to set:

$$\begin{aligned} r &= x_1 - x_2 \\ R &= x_1 + x_2 \end{aligned}$$

The Jacobian of the transformation is  $\frac{1}{2}$ . The product of the 4 wave functions has an exponent of the form:

$$\begin{aligned} (x_1 - \epsilon_1 a)^2 + (x_2 - \epsilon_2 a)^2 + (x_1 - \epsilon_3 a)^2 + (x_2 - \epsilon_4 a)^2 &= \\ r^2 + R^2 + 4a^2 - 2arp - 2aRq &= \\ (r - ap)^2 - a^2 p^2 + 4a^2 + (R - aq)^2 - a^2 q^2 & \end{aligned}$$

where  $2q = \epsilon_1 + \epsilon_2 + \epsilon_3 + \epsilon_4$  and  $2p = \epsilon_1 - \epsilon_2 + \epsilon_3 - \epsilon_4$ . Note that both  $q$  and  $p$  take values  $-2, -1, 0, 1, 2$ .

The integral becomes:

$$\frac{1}{2} e^{-\frac{a^2}{\xi^2} \left(1 - \frac{q^2 + p^2}{4}\right)} \frac{1}{\sqrt{2\pi\xi}} \int dR e^{-\frac{(R-aq)^2}{4\xi^2}} \frac{1}{\sqrt{2\pi\xi}} \int dr V(|r|) e^{-\frac{(r-ap)^2}{4\xi^2}}$$

The  $R$  integration is the first we can do and it gives a factor of  $\sqrt{2}$ . Now we are left with the integral:

$$e^{-\frac{a^2}{\xi^2} \left(1 - \frac{q^2 + p^2}{4}\right)} \frac{a}{2\sqrt{\pi\xi}} \int dr V(a|r|) e^{-\frac{a^2(r-p)^2}{4\xi^2}}$$

where we set  $r \rightarrow ar$ . Note that if  $V$  was 1 the  $r$  integral would just give us one and only the exponent would survive. Also we note that if  $p < 0$  we can make a transformation  $r \rightarrow -r$  to absorb

the extra sign. Therefore only the absolute value of  $p = 0, 1, 2$  matters. Finally after a little algebra we can show that  $4p^2 + 4q^2 = 8 + 4(\epsilon_1\epsilon_3 + \epsilon_2\epsilon_4)$ . We can write our final result:

$$e^{-\frac{a^2}{2\xi^2}\left(1 - \frac{\epsilon_1\epsilon_3 + \epsilon_2\epsilon_4}{2}\right)} \frac{a}{2\sqrt{\pi\xi}} \int dr V(a|r) e^{-\frac{a^2(r-|p|)^2}{4\xi^2}}$$

Here are the forms of the matrices:

$$|p| = \begin{pmatrix} 0 & 1 & 1 & 0 \\ 1 & 2 & 0 & 1 \\ 1 & 0 & 2 & 1 \\ 0 & 1 & 1 & 0 \end{pmatrix}$$

and

$$\frac{\epsilon_1\epsilon_3 + \epsilon_2\epsilon_4}{2} = \begin{pmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & -1 & 0 \\ 0 & -1 & 1 & 0 \\ -1 & 0 & 0 & 1 \end{pmatrix}$$

Since the matrix  $V$  only depends on those matrices we can tell that it has few independent elements and more specifically it has the form:

$$V = \begin{pmatrix} a & c & c & d \\ c & b & d & c \\ c & d & b & c \\ d & c & c & a \end{pmatrix}$$

So all and all it has 4 independent matrix elements. From the above integral those matrix elements are:

$$\begin{aligned} a &= \frac{a}{2\sqrt{\pi\xi}} \int dr V(a|r) e^{-\frac{a^2 r^2}{4\xi^2}} \\ b &= \frac{a}{2\sqrt{\pi\xi}} \int dr V(a|r) e^{-\frac{a^2 (r-2)^2}{4\xi^2}} \\ c &= e^{-\frac{a^2}{2\xi^2}} \frac{a}{2\sqrt{\pi\xi}} \int dr V(a|r) e^{-\frac{a^2 (r-1)^2}{4\xi^2}} \\ d &= e^{-\frac{a^2}{\xi^2}} \frac{a}{2\sqrt{\pi\xi}} \int dr V(a|r) e^{-\frac{a^2 r^2}{4\xi^2}} \end{aligned}$$

In the basis  $|RR\rangle, |RL\rangle, |LR\rangle, |LL\rangle$  (with that order) any state (spacial part) can be represented as a vector:

$$\begin{aligned} |_{+1-2}\rangle &= \frac{1}{2} (|R\rangle_1 + |L\rangle_1) (|R\rangle_2 - |L\rangle_2) \rightarrow \left\{ \frac{1}{2}, -\frac{1}{2}, \frac{1}{2}, -\frac{1}{2} \right\} \\ |_{-1+2}\rangle &= \frac{1}{2} (|R\rangle_1 - |L\rangle_1) (|R\rangle_2 + |L\rangle_2) \rightarrow \left\{ \frac{1}{2}, \frac{1}{2}, -\frac{1}{2}, -\frac{1}{2} \right\} \end{aligned}$$

Also we need the normalization constants:

$$\begin{aligned} \langle + - | + - \rangle &= \langle + | + \rangle \langle - | - \rangle = (1+l)(1-l) = 1-l^2 \\ \langle - + | - + \rangle &= \langle + | + \rangle \langle - | - \rangle = (1+l)(1-l) = 1-l^2 \\ \langle + + | + + \rangle &= \langle + | + \rangle^2 = (1+l)^2 \\ \langle - - | - - \rangle &= \langle - | - \rangle^2 = (1-l)^2 \end{aligned}$$

We see that either of the states  $|+-\rangle, |-+\rangle$  needs to be multiplied by a  $\frac{1}{\sqrt{1-l^2}}$  in order to be normalized to unity. Taking this into account we can evaluate using the matrix form of  $V$  and those states the Coulomb and exchange integrals:

$$U_{Coulomb} = \frac{1}{1-l^2} \int dx_1 dx_2 \langle +1 -2 | V | +1 -2 \rangle = \frac{1}{1-l^2} \frac{1}{2} \begin{pmatrix} 1 & -1 & 1 & -1 \end{pmatrix} \begin{pmatrix} a & c & c & d \\ c & b & d & c \\ c & d & b & c \\ d & c & c & a \end{pmatrix} \begin{pmatrix} 1 \\ -1 \\ 1 \\ -1 \end{pmatrix}$$

$$U_{Exchange} = \frac{1}{1-l^2} \int dx_1 dx_2 \langle -1 +2 | V | +1 -2 \rangle = \frac{1}{1-l^2} \frac{1}{2} \begin{pmatrix} 1 & 1 & -1 & -1 \end{pmatrix} \begin{pmatrix} a & c & c & d \\ c & b & d & c \\ c & d & b & c \\ d & c & c & a \end{pmatrix} \begin{pmatrix} 1 \\ -1 \\ 1 \\ -1 \end{pmatrix}$$

Carrying out the matrix multiplications gives:

$$U_C = \frac{1}{1-l^2} \left( \frac{a+b}{2} - d \right)$$

$$U_E = \frac{1}{1-l^2} \frac{a-b}{2}$$

(5)

We are going to express the Hamiltonian in the following basis:

$$|P\rangle = |++\rangle \rightarrow \frac{1}{2(1+l)} \{1, 1, 1, 1\}$$

$$|M\rangle = |--\rangle \rightarrow \frac{1}{2(1-l)} \{1, -1, -1, 1\}$$

$$|S\rangle = \frac{1}{\sqrt{2}} |+-\rangle + \frac{1}{\sqrt{2}} |-+\rangle \rightarrow \frac{1}{\sqrt{2(1-l^2)}} \{1, 0, 0, -1\}$$

$$|A\rangle = \frac{1}{\sqrt{2}} |+-\rangle - \frac{1}{\sqrt{2}} |-+\rangle \rightarrow \frac{1}{\sqrt{2(1-l^2)}} \{0, -1, 1, 0\}$$

where the vectors express the states in the  $|RR\rangle, |RL\rangle, |LR\rangle, |LL\rangle$  basis. The two parts of the Hamiltonian become:

$$H_1 + H_2 = \begin{pmatrix} 2E_+ & 0 & 0 & 0 \\ 0 & 2E_- & 0 & 0 \\ 0 & 0 & E_+ + E_- & 0 \\ 0 & 0 & 0 & E_+ + E_- \end{pmatrix}$$

$$V = \begin{pmatrix} \frac{a+b+4c+2d}{2(l+1)^2} & \frac{a-b}{2(1-l^2)} & 0 & 0 \\ \frac{a-b}{2(1-l^2)} & \frac{a+b-4c+2d}{2(1-l)^2} & 0 & 0 \\ 0 & 0 & \frac{a-d}{1-l^2} & 0 \\ 0 & 0 & 0 & \frac{b-d}{1-l^2} \end{pmatrix}$$

Notice that the Hamiltonian is almost diagonal in this basis. We can see directly that the energy of the symmetric and antisymmetric state is:

$$\begin{aligned} E_S &= \frac{a-d}{1-l^2} = E_+ + E_- + U_C + U_E \\ E_A &= \frac{b-d}{1-l^2} = E_+ + E_- + U_C - U_E \end{aligned}$$

Diagonalizing the  $|++\rangle, |--\rangle$  block does not give so nice expressions. The matrix that we need to diagonalize has the form:

$$\begin{pmatrix} \frac{a+b+4c+2d}{2(l+1)^2} + 2E_+ & \frac{a-b}{2(1-l^2)} \\ \frac{a-b}{2(1-l^2)} & \frac{a+b-4c+2d}{2(1-l)^2} + 2E_- \end{pmatrix} = \begin{pmatrix} A & C \\ C & B \end{pmatrix}$$

The UNNORMALIZED eigenvectors are:

$$|Q_{\pm}\rangle = \left( A - B \pm \sqrt{(A-B)^2 + 4C^2} \right) |++\rangle + 2C |--\rangle$$

and the corresponding eigenenergies:

$$E_{Q_{\pm}} = \frac{1}{2} \left( A + B \pm \sqrt{(A-B)^2 + 4C^2} \right)$$

Finally as we explained in part (3), the symmetric states  $|Q_{\pm}\rangle$  and  $|S\rangle$  will be associated with an antisymmetric spin singlet and the single antisymmetric state  $|A\rangle$  will be associated with any of the three symmetric triplet spin states.

To understand the exchange constant we will consider the energy of the symmetric  $|S\rangle$  (which is associated to a singlet spin state) and the antisymmetric state  $|A\rangle$  (which is associated to a triplet). This energy can be obtained from an effective Hamiltonian which involves spin:

$$H_{eff} = (E_+ + E_- + U_C) - \frac{U_E}{2} \left( 1 + \frac{4}{\hbar^2} \vec{S}_1 \cdot \vec{S}_2 \right)$$

Remember that:

$$\frac{1}{2} + \frac{2}{\hbar^2} \vec{S}_1 \cdot \vec{S}_2 = \frac{1}{2} + \frac{1}{\hbar^2} \left[ (\vec{S}_1 + \vec{S}_2)^2 - \vec{S}_1^2 - \vec{S}_2^2 \right] = \frac{1}{\hbar^2} (\vec{S}_1 + \vec{S}_2)^2 - 1$$

If the total spin (quantum number) is  $S = 1$  this quantity is  $+1$  and if  $S = 0$  it is  $-1$ . So if the two particles are in a triplet spin state which means that their spacial wave function is antisymmetric the  $U_E$  term comes with a minus sign. If the particles are in a singlet it comes with a plus sign. The exchange coefficient  $-J$  (with a minus sign) is the coefficient in front of  $\vec{S}_1 \cdot \vec{S}_2$  and it is clearly:

$$J = \frac{2}{\hbar^2} U_E = \frac{1}{\hbar^2} \frac{a-b}{1-l^2}$$

### 3 Spin- $\frac{1}{2}$ Ferromagnet

(1)

At each site there are 2 states. The whole system has  $2^N$  states.

(2)

Consider the expression:

$$\vec{\sigma}(i) \cdot \vec{\sigma}(j) = \sigma_z(i)\sigma_z(j) + 2\sigma_+(i)\sigma_-(j) + 2\sigma_-(i)\sigma_+(j) \quad (1)$$

which is obtained by using the equations  $\sigma_{\pm} = \frac{\sigma_x \pm i\sigma_y}{2}$ . Clearly if we apply a term like this to the given state all the terms containing  $\sigma_+$  will vanish. Therefore only the  $\sigma_z(i)\sigma_z(j)$  term will survive and on this state it will just give a +1. Therefore the state is an eigenstate of the Hamiltonian. To find the eigenenergy replace  $\vec{\sigma}(i) \cdot \vec{\sigma}(i+1)$  with +1 to get  $-JN$ . We can understand that intuitively: for each pair of neighboring spins that are parallel there is an energy of  $-J$ . If all of them are parallel the energy is  $-NJ$ .

(3)

At first lets try to evaluate the commutator of the Hamiltonian with  $\sigma_a^{total} = \sum_{j=1}^N \sigma_a(j)$ :

$$\begin{aligned} [H, \sigma_a^{total}] &= \sum_{i=1}^N \sum_{b=1}^3 \sum_{j=1}^N [\sigma_b(i)\sigma_b(i+1), \sigma_a(j)] \\ &= \sum_{i=1}^N \sum_{b=1}^3 \sum_{j=1}^N \{ \sigma_b(i) [\sigma_b(i+1), \sigma_a(j)] + [\sigma_b(i), \sigma_a(j)] \sigma_b(i+1) \} \\ &= \sum_{i=1}^N \sum_{b=1}^3 \sum_{j=1}^N \{ \delta_{i+1,j} \sigma_b(i) [\sigma_b(i+1), \sigma_a(i+1)] + \delta_{ij} [\sigma_b(i), \sigma_a(i)] \sigma_b(i+1) \} \\ &= \sum_{i=1}^N \sum_{c,b=1}^3 \{ \sigma_b(i) 2i\epsilon_{bac}\sigma_c(i+1) + 2i\epsilon_{bac}\sigma_c(i)\sigma_b(i+1) \} \\ &= 2i \sum_{i=1}^N \sum_{c,b=1}^3 \{ \epsilon_{bac}\sigma_b(i)\sigma_c(i+1) + \epsilon_{cab}\sigma_b(i)\sigma_c(i+1) \} \\ &= 0 \end{aligned}$$

In the penultimate line we interchanged the dummy variables  $b$  and  $c$  in the second term and then we used that  $\epsilon_{cab} = -\epsilon_{bac}$ . We see that all the components of the total spin  $\sigma_a^{total}$  commutes with the Hamiltonian. Therefore we can label the eigenstates of the Hamiltonian by the value of  $\sigma_a^{total}$  (for one particular  $a$ ). We conclude that the total angular momentum  $\sum_a \sigma_a^{total} \sigma_a^{total}$  also commutes with the Hamiltonian.

Proving that  $\sigma_a^{total} = \sum_{j=1}^N \sigma_a(j)$  commutes with  $\sum_a \sigma_a^{total} \sigma_a^{total}$  goes as follows: First we find the commutation of  $\sigma_a^{total}$  with each other:

$$[\sigma_a^{total}, \sigma_b^{total}] = \sum_{i,j=1}^N [\sigma_a(i), \sigma_b(j)] = \sum_{i,j=1}^N 2\epsilon_{abc}\delta_{ij}\sigma_c(i) = 2\epsilon_{abc}\sigma_c^{total}$$

Not surprisingly the  $\sigma_a^{total}$  satisfy the Pauli matrix commutators. Therefore they must commute with  $\sum_a \sigma_a^{total} \sigma_a^{total}$ .

$$\begin{aligned} \left[ \sum_a \sigma_a^{total} \sigma_a^{total}, \sigma_b^{total} \right] &= \sum_a \sigma_a^{total} [\sigma_a^{total}, \sigma_b^{total}] + [\sigma_a^{total}, \sigma_b^{total}] \sigma_a^{total} \\ &= 2 \sum_a \sigma_a^{total} \epsilon_{abc} \sigma_c^{total} + \sigma_c^{total} \sigma_a^{total} \epsilon_{abc} \sigma_c^{total} = 0 \end{aligned}$$

because  $\epsilon_{abc}$  which is antisymmetric under  $c \leftrightarrow a$  is multiplied with something symmetric. Clearly  $\sum_a \sigma_a^{total} \sigma_a^{total}$  is identified as the total spin operator and  $\sigma_a^{total}$  as the spin components. Hamiltonian commutes with them because it conserves the total spin.

Therefore the eigenstates of the Hamiltonian are also eigenstates of the total spin of  $N$  electrons.

Now lets calculate the ground state energy. Consider the operator

$$\begin{aligned} \vec{\sigma}(i) \cdot \vec{\sigma}(i+1) &= 4\vec{S}(i) \cdot \vec{S}(i+1) \\ &= 2 \frac{1}{\hbar^2} \left( (\vec{S}(i) + \vec{S}(i+1))^2 - \vec{S}^2(i) - \vec{S}^2(i+1) \right) \\ &= 2 \frac{1}{\hbar^2} (\vec{S}(i) + \vec{S}(i+1))^2 - 3 \end{aligned}$$

So if the sites  $i$  and  $i+1$  are in a total spin state of  $s=0$  this operator will give  $-3$  and if they are in an  $s=1$  state it will give  $+1$ . We can therefore always write:

$$\vec{\sigma}(i) \cdot \vec{\sigma}(i+1) \leq 1$$

This means that if we calculate the expectation value of this operator with some many body state  $|\Psi\rangle = |s_1 s_2, \dots, s_N\rangle$  where  $s_i = \pm 1$  we will have the inequality:

$$\langle \Psi | \vec{\sigma}(i) \cdot \vec{\sigma}(i+1) | \Psi \rangle \leq +1$$

Using this we get that:

$$\langle \Psi | H | \Psi \rangle \geq -J \sum_{i=1}^N 1 = -JN$$

As we showed in part (2) there is one state with energy exactly  $-JN$  which is the ground state energy.

The total spin of  $N$  electrons takes a maximum value of  $S = N/2$ . We will now show that all the states with the maximum spin are ground states of the Hamiltonian. First the state  $|\frac{N}{2}, \frac{N}{2}\rangle = |+_1 +_2, \dots, +_N\rangle$  with all spins in the spin-up state is a ground state as we discussed in part (2). To get all the states of maximum total spin we apply the ladder operator  $\sigma_-^{total} = \frac{1}{2}\sigma_x^{total} - i\frac{1}{2}\sigma_y^{total}$  which commutes with the Hamiltonian:

$$H \sigma_-^{total} \left| \frac{N}{2}, m \right\rangle = \sigma_-^{total} H \left| \frac{N}{2}, m \right\rangle = -JN \sigma_-^{total} \left| \frac{N}{2}, m \right\rangle$$

This means that if the state  $|\frac{N}{2}, m\rangle$  is the ground state so is the  $\sigma_-^{total} |\frac{N}{2}, m\rangle$ . Since for  $m = N/2$  we have the ground state, we have shown inductively that all the states of maximum spin are ground states of the Hamiltonian. There are  $2\frac{N}{2} + 1 = N + 1$  states.

(4)

Lets calculate some commutators:

$$\begin{aligned} [\sigma_z(i)\sigma_z(i+1), \sigma_-(j)] &= -2\delta_{ij}\sigma_-(j)\sigma_z(j+1) - 2\delta_{i+1,j}\sigma_-(j)\sigma_z(j-1) \\ [\sigma_+(i)\sigma_-(i+1), \sigma_-(j)] &= \delta_{ij}\sigma_z(j)\sigma_-(j+1) \\ [\sigma_-(i)\sigma_+(i+1), \sigma_-(j)] &= \delta_{i+1,j}\sigma_-(j-1)\sigma_z(j) \end{aligned}$$

From this we get the commutator with the Hamiltonian:

$$[H, \sigma_-(j)] = (-J)(-2\sigma_-(j)\sigma_z(j+1) - 2\sigma_-(j)\sigma_z(j-1) + 2\sigma_z(j)\sigma_-(j+1) + 2\sigma_-(j-1)\sigma_z(j))$$

If we apply this on the ground state we will get:

$$\frac{1}{J}[H, \sigma_-(j)] |G\rangle = -J(2\sigma_-(j+1) - 4\sigma_-(j) + 2\sigma_-(j-1)) |G\rangle$$

Finally multiplying by  $\frac{1}{\sqrt{N}}e^{ipj}$  and summing over  $j$  will give:

$$\begin{aligned} [H, \sum_{j=1}^N \frac{1}{\sqrt{N}} e^{ipj} \sigma_-(j)] |G\rangle &= -J \sum_{j=1}^N \frac{1}{\sqrt{N}} e^{ipj} (2\sigma_-(j+1) - 4\sigma_-(j) + 2\sigma_-(j-1)) |G\rangle \\ H |p\rangle - E_G |p\rangle &= 8J \sin^2 \frac{p}{2} |p\rangle \end{aligned}$$

And we can see that this state is an eigen state of the Hamiltonian with energy  $E_G + 8J \sin^2 \frac{p}{2}$ . The state is a superposition of states with a spin flipped and phase proportional to the position of the spin flip. Translation by a lattice site will result in a phase factors  $e^{ip}$  which means that this excited states has a definite momentum.

## 4 A free fermion system

(1)

The Hamiltonian of the system is  $H = \frac{\hat{p}^2}{2M}$ . The single particle states are:

$$\psi_\sigma(x) = \frac{1}{\sqrt{L}} e^{ikx}$$

The wavevector  $k$  satisfies  $e^{ikL} = 1 \Rightarrow k = \frac{2\pi n}{L}$ . The corresponding eigenenergy is  $E_n = \frac{\hbar^2}{2M} \frac{4\pi^2 n^2}{L^2}$ . Notice all the energy states apart from  $n = 0$  are 2-fold degenerate because  $E_n = E_{-n}$ . Including the spin this degeneracy is 4-fold.

(2)

In position space

$$\begin{aligned}
H &= \sum_{\sigma, \sigma'} \int dx dx' a_{\sigma}^{\dagger}(x) \langle x\sigma | H | x'\sigma' \rangle a_{\sigma'}(x') \\
&= \sum_{\sigma, \sigma'} \int dx dx' a_{\sigma}^{\dagger}(x) \frac{-\hbar^2}{2M} \frac{d^2}{dx^2} \delta_{\sigma\sigma'} \langle x|x' \rangle a_{\sigma'}(x') \\
&\quad \sum_{\sigma} \int dx a^{\dagger}(x) \frac{-\hbar^2}{2M} \frac{d^2}{dx^2} a(x)
\end{aligned}$$

because  $\langle x|x' \rangle = \delta(x - x')$ . Similarly in momentum space:

$$\begin{aligned}
H &= \sum_{\sigma, \sigma'} \int \frac{dp dp'}{(2\pi\hbar)^2} a_{\sigma}^{\dagger}(p) \langle p\sigma | H | p'\sigma' \rangle a_{\sigma'}(p') \\
&= \sum_{\sigma, \sigma'} \int \frac{dp dp'}{(2\pi\hbar)^2} a_{\sigma}^{\dagger}(p) \frac{p^2}{2M} \langle p|p' \rangle \delta_{\sigma\sigma'} a_{\sigma'}(p') \\
&\quad \sum_{\sigma} \int \frac{dp}{2\pi\hbar} a_{\sigma}^{\dagger}(p) \frac{p^2}{2M} a_{\sigma}(p)
\end{aligned}$$

again  $\langle p|p' \rangle = 2\pi\hbar\delta(p - p')$ .

**(3)**

Follow the derivation in the Lecture notes, pages 247-248.

**(4)**

Because  $N/2$  is odd we can write  $N = 4K + 2$  for some integer  $K$ . The ground state is constructed by adding fermions to states with increasing energy. First the ground state  $|n = 0, \sigma = \uparrow, \downarrow\rangle$  will be filled with two electrons and  $N - 2 = 4K$  electrons will remain. Next we observe that the states with opposite quantum number  $n$  have the same energy. Including spin all the states have 4-fold degeneracy. The next energy levels to be filled are the 4 states  $|n = \pm 1, \sigma = \uparrow, \downarrow\rangle$ , then  $|n = \pm 2, \sigma = \uparrow, \downarrow\rangle \dots |n = \pm K, \sigma = \uparrow, \downarrow\rangle$ . The most energetic state to be filled has energy quantum number  $n = K = \pm \frac{N-2}{4}$ . The ground state can be expressed using creation operators:

$$|gnd\rangle = a_{0\uparrow}^{\dagger} a_{0\downarrow}^{\dagger} \prod_{i=1}^{\frac{N-2}{4}} a_{i\uparrow}^{\dagger} a_{-i\uparrow}^{\dagger} a_{i\downarrow}^{\dagger} a_{-i\downarrow}^{\dagger} |vacuum\rangle$$

The Fermi energy corresponding to those  $N$  electrons is:

$$E_F = E_{\frac{N-2}{4}} = \frac{\hbar^2}{2M} \frac{4\pi^2}{L^2} \frac{(N-2)^2}{16}$$

(5)

Look at Lecture Notes, page 243 (2nd quantization). The Slatter determinant occurs as the “matrix element” between the antisymmetrized states of  $M$  fermions:

$$\begin{aligned} |X\rangle &= \frac{1}{\sqrt{N!}} \sum_P \text{sign}(P) |x_{P_1}\rangle |x_{P_2}\rangle \dots |x_{P_M}\rangle \\ |\Psi\rangle &= \frac{1}{\sqrt{N!}} \sum_P \text{sign}(Q) |\psi_{Q_1}\rangle |\psi_{Q_2}\rangle \dots |\psi_{Q_M}\rangle \end{aligned}$$

where  $\text{sign}(P)$  is +1 for even permutations and -1 for odd permutations. The matrix element is:

$$\begin{aligned} \langle X|\Psi\rangle &= \frac{1}{N!} \sum_{P,Q} \text{sign}(P)\text{sign}(Q) \langle x_{P_1}|\psi_{Q_1}\rangle \langle x_{P_2}|\psi_{Q_2}\rangle \dots \langle x_{P_M}|\psi_{Q_M}\rangle \\ &= \frac{1}{N!} \sum_{P,Q} \text{sign}(P)\text{sign}(Q) \langle x_1|\psi_{(QP^{-1})_1}\rangle \langle x_2|\psi_{(QP^{-1})_2}\rangle \dots \langle x_M|\psi_{(QP^{-1})_M}\rangle \\ &= \frac{1}{N!} \sum_{P,R} \text{sign}(R) \langle x_1|\psi_{R_1}\rangle \langle x_2|\psi_{R_2}\rangle \dots \langle x_M|\psi_{R_M}\rangle \\ &= \sum_R \text{sign}(R) \langle x_1|\psi_{R_1}\rangle \langle x_2|\psi_{R_2}\rangle \dots \langle x_M|\psi_{R_M}\rangle \end{aligned}$$

In the second line we sort the terms of the  $P$  summation which will result in the terms of the  $Q$  summation to come in  $R = QP^{-1}$  order. A permutation can be inverted if one sorts its elements. The arrangement of the original positions of the elements in the sorted list gives  $P^{-1}$ . Also two permutations can be composed if one permutes the elements of the one according to the order defined by the other. Combining permutations multiplies their characteristic sign and therefore:

$$\text{sign}(P)\text{sign}(Q) = \text{sign}(P^{-1})\text{sign}(Q) = \text{sign}(QP^{-1}) = \text{sign}(R)$$

This is done in the third line. In the 4th line we just sum  $P$  away, getting an extra factor of  $N!$  because nothing depends on it anymore. The result is identical to the definition of the determinant of a matrix with elements  $\langle x_i|\psi_j\rangle = \psi_j(x_i)$  and therefore:

$$\langle X|\Psi\rangle = \begin{vmatrix} \psi_1(x_1) & \psi_2(x_1) & \dots & \psi_M(x_1) \\ \psi_1(x_2) & \psi_2(x_2) & \dots & \psi_M(x_2) \\ \vdots & \vdots & & \vdots \\ \psi_1(x_M) & \psi_2(x_M) & \dots & \psi_M(x_M) \end{vmatrix}$$

This  $M \times M$  determinant is the Slatter determinant. So far I used a simplified notation that does not involve spins. To return to the original notation, every odd index corresponds to spin up and every even index to spin down states, more specifically  $|x_1x_2x_3x_4 \dots x_{M-1}x_M\rangle \rightarrow |x_{1\uparrow}x_{1\downarrow}x_{2\uparrow}x_{2\downarrow} \dots x_{\frac{N}{2}\uparrow}x_{\frac{N}{2}\downarrow}\rangle$ . Similarly the  $|\psi_1\psi_2\psi_3\psi_4 \dots \psi_{M-1}\psi_M\rangle \rightarrow |\psi_{1\uparrow}\psi_{1\downarrow}\psi_{2\uparrow}\psi_{2\downarrow} \dots \psi_{\frac{N}{2}\uparrow}\psi_{\frac{N}{2}\downarrow}\rangle$ . Now we will use the fact that actually  $|x_{i\sigma}\rangle = |x_i\rangle |\sigma\rangle$  and similarly  $|\psi_{i\sigma}\rangle = |\psi_i\rangle |\sigma\rangle$  this means that  $\langle x_{i\sigma}|\psi_{j\sigma'}\rangle = \delta_{\sigma\sigma'}\psi_j(x_i)$ . So half

the elements are zero:

$$\langle X|\Psi\rangle = \begin{vmatrix} \psi_1(x_{1\uparrow}) & 0 & \psi_2(x_{1\uparrow}) & 0 & \dots & \psi_{\frac{N}{2}}(x_{1\uparrow}) & 0 \\ 0 & \psi_1(x_{1\downarrow}) & 0 & \psi_2(x_{1\downarrow}) & 0 & \dots & \psi_{\frac{N}{2}}(x_{1\downarrow}) \\ \psi_1(x_{2\uparrow}) & 0 & \psi_2(x_{2\uparrow}) & 0 & \dots & \psi_{\frac{N}{2}}(x_{2\uparrow}) & 0 \\ 0 & \psi_1(x_{2\downarrow}) & 0 & \psi_2(x_{2\downarrow}) & 0 & \dots & \psi_{\frac{N}{2}}(x_{2\downarrow}) \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \psi_1(x_{\frac{N}{2}\uparrow}) & 0 & \psi_2(x_{\frac{N}{2}\uparrow}) & 0 & \dots & \psi_{\frac{N}{2}}(x_{\frac{N}{2}\uparrow}) & 0 \\ 0 & \psi_1(x_{\frac{N}{2}\downarrow}) & 0 & \psi_2(x_{\frac{N}{2}\downarrow}) & 0 & \dots & \psi_{\frac{N}{2}}(x_{\frac{N}{2}\downarrow}) \end{vmatrix}$$

This matrix is block diagonal (surprise?). To convince your self about it count the number of non zero elements in each row/column and observe that their position is the same at every row and column, in the same way that it happens to a block diagonal matrix in its normal form. Then the determinant can be factorized in a spin up and spin down part:

$$\langle X|\Psi\rangle = \begin{vmatrix} \psi_1(x_{1\uparrow}) & \psi_2(x_{1\uparrow}) & \dots & \psi_{\frac{N}{2}}(x_{1\uparrow}) \\ \psi_1(x_{2\uparrow}) & \psi_2(x_{2\uparrow}) & \dots & \psi_{\frac{N}{2}}(x_{2\uparrow}) \\ \vdots & \vdots & \vdots & \vdots \\ \psi_1(x_{\frac{N}{2}\uparrow}) & \psi_2(x_{\frac{N}{2}\uparrow}) & \dots & \psi_{\frac{N}{2}}(x_{\frac{N}{2}\uparrow}) \end{vmatrix} \begin{vmatrix} \psi_1(x_{1\downarrow}) & \psi_2(x_{1\downarrow}) & \dots & \psi_{\frac{N}{2}}(x_{1\downarrow}) \\ \psi_1(x_{2\downarrow}) & \psi_2(x_{2\downarrow}) & \dots & \psi_{\frac{N}{2}}(x_{2\downarrow}) \\ \vdots & \vdots & \vdots & \vdots \\ \psi_1(x_{\frac{N}{2}\downarrow}) & \psi_2(x_{\frac{N}{2}\downarrow}) & \dots & \psi_{\frac{N}{2}}(x_{\frac{N}{2}\downarrow}) \end{vmatrix} \quad (2)$$

If it is not exactly what you got remember that the determinant does not change when you transpose a matrix.

### (6)

We can do an expansion for each of the two determinants in 2. We will focus on the  $\uparrow$  one. There are  $N/2$  states. In the ground state the involved states have quantum numbers  $n = 0, \pm 1, \pm 2, \dots, \pm \frac{N-2}{4}$  (in total  $1 + 2\frac{N-2}{4} = N/2$  states). Lets denote  $z_k = e^{i\frac{2\pi x_k}{L}}$  so that  $\psi_n(x_j) = \frac{1}{\sqrt{L}} z_j^n$ . We will order the negative values before the positive:

$$\begin{vmatrix} \psi_1(x_{1\uparrow}) & \psi_2(x_{1\uparrow}) & \dots & \psi_{\frac{N}{2}}(x_{1\uparrow}) \\ \psi_1(x_{2\uparrow}) & \psi_2(x_{2\uparrow}) & \dots & \psi_{\frac{N}{2}}(x_{2\uparrow}) \\ \vdots & \vdots & \vdots & \vdots \\ \psi_1(x_{\frac{N}{2}\uparrow}) & \psi_2(x_{\frac{N}{2}\uparrow}) & \dots & \psi_{\frac{N}{2}}(x_{\frac{N}{2}\uparrow}) \end{vmatrix} = \frac{1}{\sqrt{L^{N/2}}} \begin{vmatrix} z_1^{-\frac{N-2}{4}} & \dots & 1 & \dots & z_1^{\frac{N-2}{4}} \\ z_2^{-\frac{N-2}{4}} & \dots & 1 & \dots & z_2^{\frac{N-2}{4}} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ z_{N/2}^{-\frac{N-2}{4}} & \dots & 1 & \dots & z_{N/2}^{\frac{N-2}{4}} \end{vmatrix} = \frac{1}{\sqrt{L^{N/2}}} \prod_{k=1}^{N/2} z_k^{-\frac{N-2}{4}} \prod_{i<j=1}^{N/2} (z_i - z_j)$$

One can also show (with a nice graphical argument) that this determinant can written:

$$\frac{(-1)^{\frac{N-2}{4}}}{\sqrt{L^{N/2}}} \prod_{i=1}^{\frac{N}{2}} \prod_{k=1}^{\frac{N-2}{4}} \left(1 - z_i z_{(i+2k) \bmod (N/2)}^{-1}\right) \quad (3)$$

the factor  $(i + 2k) \bmod(\frac{N}{2})$  means to evaluate  $i + 2k$  with modulo  $N/2$ . What is significant about this expression is that it does not contain any prefactors and also everything depends on differences of coordinates of different particles as it is appropriate for a center of mass wave function (zero linear total momentum as we are having). This means that an arbitrary translation of all the particles will leave this wave function invariant. For example for  $N/2 = 5$  and  $(N - 2)/4 = 2$  this expression would give:

$$(1 - z_1 z_3^{-1})(1 - z_2 z_4^{-1})(1 - z_3 z_5^{-1})(1 - z_4 z_1^{-1})(1 - z_5 z_2^{-1}) \times \\ (1 - z_1 z_5^{-1})(1 - z_2 z_1^{-1})(1 - z_3 z_2^{-1})(1 - z_4 z_3^{-1})(1 - z_5 z_4^{-1})$$

which, after you cancel some minus signs is the Vandermonde expansion with the prefactor included.

Combining the two results for spin up and spin down states will give:

$$\langle X | \Psi \rangle = \frac{1}{L^{N/2}} e^{-i \frac{N-2}{4} \frac{2\pi}{L} N X_{CM}} \prod_{a < b=1}^{N/2} (e^{i \frac{2\pi}{L} x_{a\uparrow}} - e^{i \frac{2\pi}{L} x_{b\uparrow}})(e^{i \frac{2\pi}{L} x_{a\downarrow}} - e^{i \frac{2\pi}{L} x_{b\downarrow}})$$

where  $X_{CM} = \frac{1}{N} \sum_{k=1}^{N/2} (x_{k\uparrow} + x_{k\downarrow})$  is the center of mass. Note that there is a *TYPO IN THE GIVEN SOLUTION*. There is a prefactor (apart from the normalization factor). This prefactor is absolutely necessary to guarantee that the total wave function has zero total linear momentum. To demonstrate that we can apply the operator  $\exp(-i\hat{P}a/\hbar)$  where  $\hat{P} = \sum_{\sigma,i} \hat{p}_{i,\sigma}$  is the total momentum of all the particles of all energies and spins. As we know this operator will translate every electron by the same distance  $a$  and  $e^{i \frac{2\pi}{L} x_{a\uparrow}} \rightarrow e^{i \frac{2\pi}{L} a} e^{i \frac{2\pi}{L} x_{a\uparrow}}$ . Collecting all the  $a$  dependent factors will give a factor of

$$e^{-i \frac{N-2}{4} \frac{2\pi}{L} 2Na} \left( e^{i \frac{2\pi}{L} a} \right)^{\frac{1}{2} \frac{N}{2} (\frac{N}{2} - 1)} = 1$$

This means that the wave function is an eigen state of the total momentum with  $P = 0$ . The total momentum is zero. This is consistent with the expression 3 where only the differences of coordinates appear.

(7)

What we do is swap the creation and annihilation operators for the states that are below the Fermi energy.

$$b_{n,\sigma} = a_{n,\sigma}^\dagger \\ b_{n,\sigma}^\dagger = a_{n,\sigma}$$

but only for  $E_n < E_F$ . For energies higher than the fermi energy we retain the original definitions. The operator  $b_{n,\sigma}$  will annihilate the ground state because it will try to create a particle in a state that is already occupied. Similarly the  $a_{n,\sigma}$  for  $E_n > E_F$  will also destroy the ground state because it will annihilate an empty state. The hamiltonian will be broken into two parts: one with states above the Fermi energy and another with states below:

$$H = \sum_{n\sigma} E_n a_{n\sigma}^\dagger a_{n\sigma} = \sum_{n\sigma} [\theta(E_n - E_F) a_{n\sigma}^\dagger a_{n\sigma} + \theta(E_F - E_n) b_{n\sigma} b_{n\sigma}^\dagger]$$

$$\begin{aligned}
&= \sum_{n\sigma} [\theta(E_n - E_F) E_n a_{n\sigma}^\dagger a_{n\sigma} + \theta(E_F - E_n) E_n (1 - b_{n\sigma}^\dagger b_{n\sigma})] \\
&= E_G + \sum_{n\sigma} E_n [\theta(E_n - E_F) a_{n\sigma}^\dagger a_{n\sigma} - \theta(E_F - E_n) b_{n\sigma}^\dagger b_{n\sigma}]
\end{aligned}$$

where,  $\theta(x)$  is the step function (+1 if  $x > 0$  and zero for  $x < 0$ ) and  $\sum_{n\sigma} \theta(E_F - E_n) E_n = E_G$  is the sum of the energies that are below the Fermi energy which by definition is the ground state energy. From this expression we see that we have positive energy electronic excitations and negative energy hole excitations. Both the electron and the hole states are 4-fold degenerate as before: the energy is independent of both the spin and the sign of the energy quantum number.

(8)

1. Two particles:  $a_\sigma^\dagger(p) a_{\sigma'}^\dagger(p') |gnd\rangle$  with energy  $\frac{p^2+p'^2}{2M}$  the momentum is not exactly arbitrary: the single particle energy must be above the fermi energy so  $p, p' > p_F$ . The total number of particles is  $N + 2$ .
2. Two holes:  $b_\sigma^\dagger(p) b_{\sigma'}^\dagger(p') |gnd\rangle = a_\sigma(p) a_{\sigma'}(p') |gnd\rangle$  with energy  $-\frac{p^2+p'^2}{2M}$ . There are two particles missing compared to the ground state and this is why the energy is lower. The total number of particles is  $N - 2$ . The momenta must be  $p, p' < p_F$ .
3. One particle and one hole:  $b_\sigma^\dagger(p) a_{\sigma'}^\dagger(p') |gnd\rangle = a_\sigma(p) a_{\sigma'}^\dagger(p') |gnd\rangle$  a particle is removed below the fermi energy and another one is created above. The total number of particles is  $N$  and the momenta are:  $p < p_F < p'$ .