

Physics 581: Quantum Mechanics II
Department of Physics, UIUC
Spring Semester 2007
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Problem Set No. 3:
Spin and Angular Momentum
Due Date: Monday 3/12/2007

1 Time evolution of a spin in an external magnetic field and Spin Resonance

In this problem we will consider the time evolution of a single spin $S = 1/2$ coupled to an external magnetic field. We will assume that the center of mass is fixed and hence translational degrees of freedom will not be considered at all. The Hamiltonian for this system is just the Zeeman interaction

$$\hat{H} = -\frac{ge}{2mc}\hat{S} \cdot \vec{B}(t)$$

where we will assume that the magnetic field is a sum of a static and a time-dependent periodic components:

$$\vec{B}(t) = B_0\vec{e}_z + B_1 \cos(\omega t)\vec{e}_x - B_1 \sin(\omega t)\vec{e}_y$$

where \vec{e}_x , \vec{e}_y and \vec{e}_z are three orthonormal unit vectors. We will assume that $B_0 \gg B_1$.

Let $|\psi(t)\rangle$ be the ket for a state in the Schrödinger picture.

1. Derive the equation of motion for the spin operator \hat{S} in the Heisenberg representation.
2. Consider first the case $B_1 = 0$, *i. e.* the time-dependent piece of the magnetic field is absent. Construct for this case the unitary time evolution operator $\hat{U}(t)$ for states $|\psi(t)\rangle$ in the Schrödinger picture. Find a relation between $\hat{U}(t)$ and rotation operators in spin space. What is the equivalent rotation? Give a physical justification for your answer.
3. Consider again the case $B_1 = 0$ and assume that the initial state is

$$|\psi(0)\rangle = |\vec{n}, \uparrow\rangle = \begin{pmatrix} \cos(\theta/2)e^{-i\phi/2} \\ \sin(\theta/2)e^{i\phi/2} \end{pmatrix}$$

Find the vector \vec{n} for which this is an eigenstate of $\hat{S} \cdot \vec{n}$ with eigenvalue $+\hbar/2$. Use the time evolution operator you derived above to find the evolved state at time t . How does the vector \vec{n} evolve with time?

4. You will consider now the case $B_1 \neq 0$. Assume that the initial state is $|\psi(0)\rangle$ the spin is up. Consider now $|\psi_\omega(t)\rangle$, the rotated state of the Schrödinger state $|\psi(t)\rangle$, defined by

$$|\psi_\omega(t)\rangle = e^{-i\omega t \hat{S}_z / \hbar} |\psi(t)\rangle$$

where ω is the angular frequency of the time-dependent field. Derive the Schrödinger Equation obeyed by $|\psi_\omega(t)\rangle$. Find the operator $\hat{U}_\omega(t)$ which generates the time evolution in the rotating frame, *i. e.* $|\psi_\omega(t)\rangle = \hat{U}_\omega(t) |\psi_\omega(0)\rangle$.

5. Rotate the state $|\psi_\omega(t)\rangle$ back to the Lab frame and show that in the \hat{S}_z basis the resulting state $|\psi(t)\rangle$ at time t is

$$|\psi(t)\rangle = \begin{pmatrix} \left[\cos\left(\frac{\Omega t}{2}\right) + i \frac{\omega_0 - \omega}{\Omega} \sin\left(\frac{\Omega t}{2}\right) \right] e^{i\omega t/2} \\ i \frac{gB_1}{\Omega} \sin\left(\frac{\Omega t}{2}\right) e^{-i\omega t/2} \end{pmatrix}$$

where

$$\omega_0 = \frac{geB_0}{mc} \quad \Omega = \omega_0 \left[\left(\frac{B_1}{B_0}\right)^2 + \left(1 - \frac{\omega}{\omega_0}\right)^2 \right]$$

6. Consider the resonant case $\omega = \omega_0$. How long do we have to wait for the spin to flip?

2 Charged particle on a ring as a two level system

Consider an electron of charge e and mass M moving on a metallic ring of radius R of negligible thickness. The electron is free to move inside the ring. We will use a coordinate system with its origin located at the center of the ring, which lies on the xy plane. The ring is placed in a uniform magnetic field $\vec{B} = B\hat{z}$, perpendicular to the ring. In addition, there is an electric field $\vec{E} = E(\cos\varphi, \sin\varphi, 0)$, on the xy plane. Here φ is the azimuthal angle and will serve as the coordinate of the electron on the ring. We will work in the circular gauge in which the vector potential $\vec{A} = (A_r, A_\varphi, A_z) = (0, A_\varphi, 0)$ and $A_\varphi = \Phi/2\pi R$, where Φ is the flux of the magnetic field \vec{B} flowing through the ring, $\Phi = \pi R^2 B$. In what follows we will use the combination $eRA_\varphi/c \equiv \Phi/\phi_0$, where $\phi_0 = hc/e$ is the flux quantum.

The Hamiltonian for the electron is

$$H = -\frac{\hbar^2}{2MR^2} \left(\frac{\partial}{\partial \varphi} - i \frac{\Phi}{\phi_0} \right)^2 - eR(E_x \cos \varphi + E_y \sin \varphi) \quad (1)$$

where $\varphi \in [0, 2\pi)$.

1. Show that the normalized eigenstates $|m\rangle$ of the Hamiltonian at zero electric field, H_0 , are eigenstates of the azimuthal angular momentum $p_\varphi = \frac{\hbar}{i} \frac{\partial}{\partial \varphi}$ and are labeled by an arbitrary integer m . In position space the eigenstates have the form $\langle \varphi | m \rangle = \frac{1}{\sqrt{2\pi}} e^{im\varphi}$. Find the exact eigenvalues (at zero electric field) E_m^0 .
2. Use the results of the previous item to show that if $\frac{\Phi}{\phi_0} = \frac{m}{2}$ ($m \in \mathbf{Z}$), the states $|m-1\rangle$ and $|m\rangle$ are degenerate. Use this result to find the ground state and its energy as a function of $\frac{\Phi}{\phi_0}$.
3. The current operator for this problem is given by

$$j_\varphi = i \frac{\phi_0}{2\pi MR} \left(\frac{\partial}{\partial \varphi} - i \frac{\Phi}{\phi_0} \right) \quad (2)$$

Show that the states $|m\rangle$ are eigenstates of j_φ and find its eigenvalues. How does the current carried by the ground state vary as a function of flux?

4. In what follows you will concentrate on values of the magnetic flux such that $\frac{\Phi}{\phi_0} \approx \frac{1}{2}$, where the states $|0\rangle$ and $|1\rangle$ become degenerate. Calculate the matrix elements of the full Hamiltonian in the two-dimensional Hilbert space spanned by these two states. Write this effective Hamiltonian, which is a 2×2 matrix, in terms of the three Pauli matrices σ_i ($i = 1, 2, 3$) and the 2×2 identity matrix I in the form

$$H_{\text{eff}} = a_0 I + a_1 \sigma_1 + a_2 \sigma_2 + a_3 \sigma_3 \quad (3)$$

How do the four coefficients depend on the magnetic flux Φ and on the components of the electric field E_x and E_y ?

5. Make an analogy with the problem of a spin- $\frac{1}{2}$ particle in an external magnetic field and reinterpret the coefficients in terms of this analogy. Using the spin analogy, define a polarization $\vec{\sigma}$. What is the meaning of the components of this operator in terms of the two nearly degenerate states of the electron on the ring? Write the polarization in the Heisenberg representation of the effective Hamiltonian and find its equation of motion. Show that the polarization exhibits a precessional motion and calculate the frequency ω_0 of this precession.
6. Imagine now the the electric field points along the x -axis but that it varies in time as $\vec{E} = \hat{e}_x E \cos \omega t$. For the purposes of this problem the electromagnetic field is classical. Consider the case in which, at time $t = 0$, the

electron is in the state $|0\rangle$. Find the probability to find the electron in state $|1\rangle$ at some arbitrary time T . Show that this probability is periodic in time and find the period T_0 . Find the times T^* for which the electron makes the transition with probability one. How is this similar to spin resonance?

3 Harmonic Oscillators and Angular Momentum

Consider a system of two harmonic oscillators defined in terms of their creation and annihilation operators $\hat{a}_1, \hat{a}_1^\dagger, \hat{a}_2$ and \hat{a}_2^\dagger . Let $|0\rangle_1$ and $|0\rangle_2$ be the ground states of the two oscillators.

1. Show that the operators

$$\hat{J}_i = \frac{\hbar}{2} \sum_{\alpha, \beta=1,2} \hat{a}_\alpha^\dagger \sigma_i^{\alpha\beta} \hat{a}_\beta \quad (4)$$

obey the angular momentum algebra if the three 2×2 matrices σ_i are the Pauli matrices.

2. Find an expression for the operator $(\vec{J})^2$ and the oscillator occupation numbers. Find the relation between the oscillator occupation numbers N_α and the eigenvalues J and m .
3. Write the kets $|j, m\rangle$ in terms of the oscillator creation and annihilation operators. Use this approach to derive the formula

$$\hat{J}^\pm = \sqrt{j(j+1) - m(m \pm 1)} |j, m \pm 1\rangle \quad (5)$$

4. Derive an expression for the matrix elements of the operators $\hat{K}^\dagger = \hat{a}_1^\dagger \hat{a}_2^\dagger$ and its adjoint \hat{K} in the angular momentum basis $\{|j, m\rangle\}$. What is the action of these operators on the states $|j, m\rangle$?

4 Addition of spin angular momenta

Consider the addition of two spin- $\frac{1}{2}$ angular momenta, $\vec{S}^{(1)}$ and $\vec{S}^{(2)}$.

1. How many states are there in the product basis?
2. If $\vec{J} = \vec{S}^{(1)} + \vec{S}^{(2)}$, what are the possible eigenvalues of $\vec{J} \cdot \vec{J}$?
3. By using the recursive algorithm construct all the Clebsch-Gordan coefficients for this problem.