1 Time evolution of a spin an an external magnetic field and Spin Resonance

(1)

$$\begin{aligned}\frac{d\hat{S}_i(t)}{dt} &= \frac{1}{i\hbar}\left[\hat{S}_i(t),\hat{H}(t)\right]\\ &= -\frac{ge}{2mc}\frac{1}{i\hbar}\sum_j B_j(t)\left[\hat{S}_i(t),\hat{S}_j(t)\right]\\ &= -\frac{ge}{2mc}\frac{1}{i\hbar}\sum_j B_j(t)i\hbar\epsilon_{ijk}\hat{S}_k(t)\end{aligned}$$

And in vector form:

$$\frac{d\vec{S}(t)}{dt} = \frac{ge}{2mc}\hat{\vec{S}}(t)\times\vec{B}(t)$$

(2)

The time evolution operator reads:

$$U(t) = e^{-it\hat{H}/\hbar} = e^{it\omega_0 S_z/\hbar} = \begin{pmatrix} e^{it\omega_0/2} & 0 \\ 0 & e^{-it\omega_0/2}\end{pmatrix}$$

where $\omega_0 = \frac{geB_0}{2mc}$. The rotation operator is given by:

$$R(t) = e^{-i\vec{\phi}\cdot\hat{\vec{S}}/\hbar}$$

where $\vec{\phi}$ is the rotation vector: its direction and magnitude are axis and the angle or rotation respectively. Comparing the two we get:

$$\vec{\phi} = -t\frac{geB_0}{2mc}\hat{z} = -\omega_0 t\hat{z}$$

The effect of the constant magnetic field is that it causes counter clockwise (if $\omega_0 > 0$) Larmor precession around the $\hat{z}$ axis with angular frequency ω_0.

(3)

The easiest way to find this $\vec{n}$ is to think as follows: $\left\langle\hat{\vec{S}}\right\rangle$ when calculated for a spin up state will give $\frac{\hbar}{2}\hat{z}$. Therefore by just evaluating the expectation value of the spin for a spin eigenstate along a certain direction, we can get this direction. Using this particular wavevector and evaluating the spin expectation value gives:

$$\frac{2}{\hbar}\left\langle\hat{\vec{S}}\right\rangle = \vec{n} = \cos\phi\sin\theta\hat{x} + \sin\phi\sin\theta\hat{y} + \cos\theta\hat{z}$$

The spin operator is:

$$\hat{\vec{S}} = \frac{\hbar}{2}\vec{\sigma}$$

and the Pauli matrices are:

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$
$$\sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$
$$\sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

From this:

$$\hat{\vec{S}} \cdot \vec{n} = \frac{\hbar}{2} \begin{pmatrix} \cos\theta & e^{-i\phi}\sin\theta \\ e^{i\phi}\sin\theta & -\cos\theta \end{pmatrix}$$

Let $\begin{pmatrix} a \\ b \end{pmatrix}$ be an eigenvector with eigenvalue $\frac{\hbar}{2}$ then:

$$a\cos\theta + be^{-i\phi}\sin\theta = a$$
$$ae^{i\phi}\sin\theta - b\cos\theta = b$$

The two equations are not independent. Bringing the all terms containing a in the same side and using $1 - \cos\theta = 2\sin^2\frac{\theta}{2}$, $1 + \cos\theta = 2\cos^2\frac{\theta}{2}$ and $\sin\theta = 2\sin\frac{\theta}{2}\cos\frac{\theta}{2}$ gives:

$$be^{-i\phi}\cos\frac{\theta}{2} = a\sin\frac{\theta}{2}$$
$$ae^{i\phi}\sin\frac{\theta}{2} = b\cos\frac{\theta}{2}$$

It is easy to demonstrate that both equations are satisfied by:

$$a = e^{-i\phi/2}\cos\frac{\theta}{2}$$
$$b = e^{i\phi/2}\sin\frac{\theta}{2}$$

and also these expressions satisfy the normalization condition $|a|^2 + |b|^2 = 1$. Applying the diagonal time evolution operator on this state gives:

$$|\psi(t)\rangle = \hat{U}(t)\,|\psi(0)\rangle = \begin{pmatrix} e^{-i(\phi-\omega_0 t)/2}\cos\frac{\theta}{2} \\ e^{i(\phi-i\omega t)/2}\sin\frac{\theta}{2} \end{pmatrix}$$

Therefore applying the time evolution operator precesses $\vec{n}$ around the $\hat{z}$ axis.

(4)

The effect of the exponential factor $e^{-i\omega t\hat{S}_z/\hbar}$ is to precess the wavevector on which it is applied with a frequency ω around the $\hat{z}$ axis. By doing this we hope to "undo" the rotation of the B_1 magnetic field.

Lets take the time derivative of the transformed wavevector:

$$i\hbar\frac{d}{dt}|\psi_\omega(t)\rangle = \omega\hat{S}_z e^{-i\omega t\hat{S}_z/\hbar}|\psi(t)\rangle + e^{-i\omega t\hat{S}_z/\hbar}\hat{H}(t)\,|\psi(t)\rangle$$
$$= (\hat{H}_\omega(t) + \omega\hat{S}_z)\,|\psi_\omega(t)\rangle$$

where the transformed Hamiltonian is:

$$\hat{H}_\omega(t) = e^{-i\omega t\hat{S}_z/\hbar}\hat{H}(t)e^{i\omega t\hat{S}_z/\hbar}$$

The B_0 part of the hamiltonian commutes with $\hat{S}_z$ and this transformation will leave it invariant. The B_1 part of the Hamiltonian in matrix form is:

$$-\omega_0\frac{B_1}{B_0}\frac{\hbar}{2}\begin{pmatrix} 0 & e^{i\omega t} \\ e^{-i\omega t} & 0 \end{pmatrix}$$

Also

$$e^{-i\omega t\hat{S}_z/\hbar} = \begin{pmatrix} e^{-i\omega t/2} & 0 \\ 0 & e^{i\omega t/2} \end{pmatrix}$$

After performing the matrix multiplications:

$$\hat{H}_\omega(t) = -\omega_0\frac{\hbar}{2}\begin{pmatrix} 1 & B_1/B_0 \\ B_1/B_0 & -1 \end{pmatrix} = -\omega_0\frac{\hbar}{2}\left(\sigma_z + \frac{B_1}{B_0}\sigma_x\right)$$

Transforming into the rotated frame deprives the Hamiltonian from the time dependence. The time evolution operator is:

$$U_\omega(t) = e^{-it(\hat{H}_\omega+\omega\hat{S}_z)/\hbar} = e^{-it\left((\omega-\omega_0)\hat{S}_z-\omega_0\hat{S}_x B_1/B_0\right)/\hbar}$$

The B_1 component of the magnetic field is frozen to its initial value.

(5)

Lets revert to the original frame:

$$|\psi(t)\rangle = e^{i\omega t\hat{S}_z/\hbar}e^{-it\left((\omega-\omega_0)\hat{S}_z-\omega_0\hat{S}_x B_1/B_0\right)/\hbar}\,|\psi(0)\rangle$$

where $|\psi(t)\rangle = |+\rangle$. The first term of the right hand side is diagonal whereas the exponent of the second term is of the form:

$$-\frac{it}{2}M = -\frac{it}{2}\begin{pmatrix} \omega-\omega_0 & -\omega_0 B_1/B_0 \\ -\omega_0 B_1/B_0 & -(\omega-\omega_0) \end{pmatrix} = -\frac{it}{2}\begin{pmatrix} a & b \\ b & -a \end{pmatrix}$$

There is a standard technique to evaluate exponents of matrices and this is through the the characteristic polynomial:

$$(a-x)(-a-x)-b^2 = 0$$

From which we get that

$$x^2 = b^2 - a^2 = \Omega^2$$

where

$$\Omega = \sqrt{b^2-a^2} = \omega_0\sqrt{\left(1-\frac{\omega}{\omega_0}\right)^2 + \left(\frac{B_1}{B_0}\right)^2}$$

Since every matrix satisfies its characteristic polynomial we can replace every even power of a matrix with a constant:

$$e^{-\frac{it}{2}M} = \cos\frac{tM}{2} - i\sin\frac{tM}{2} = \mathbf{1}\cos\frac{\Omega t}{2} - i\frac{M}{\Omega}\sin\frac{\Omega t}{2}$$

where $\mathbf{1}$ is the unit matrix. Carrying out the matrix multiplications:

$$|\psi(t)\rangle = \begin{pmatrix} \left(\cos\frac{\Omega t}{2} + i\frac{\omega_0-\omega}{\Omega}\sin\frac{\Omega t}{2}\right)e^{i\omega t/2} \\ i\frac{\omega_0}{\Omega}\frac{B_0}{B_1}\sin\frac{\Omega t}{2}e^{-i\omega t/2} \end{pmatrix}$$

(6)

In the resonant case $\Omega = \frac{B_1}{B_0}$. The probability that the system is in a down-spin state is: $\frac{\omega_0^2}{\Omega^2}\frac{B_0^2}{B_1^2}\sin^2\frac{\Omega t}{2}$. At $t = 0$ it is zero by construction. The first time that it will become 1 is after some time T:

$$T = \frac{\pi}{\Omega} = \frac{\pi B_0}{B_1}$$

2 Charged particle on a ring as a two level system

(1)

Lets consider the eigenstates of the angular momentum operator:

$$\langle\phi|m\rangle = \frac{1}{\sqrt{2\pi}}e^{im\phi}$$

The corresponding angular momentum eigenvalue is $\hbar m$. The periodic boundary conditions imply that:

$$\langle\phi + 2\pi|m\rangle = \langle\phi|m\rangle \Rightarrow e^{i2\pi m} = 1$$

and therefore m is an integer. Applying the Hamiltonian on these states will give:

$$\begin{aligned}\langle\phi|H|m\rangle &= -\frac{1}{\sqrt{2\pi}}\frac{\hbar^2}{2MR^2}\left(\frac{\partial}{\partial\phi} - i\frac{\mathbf{\Phi}}{\phi_0}\right)^2 e^{im\phi} \\ &= -\frac{1}{\sqrt{2\pi}}\frac{\hbar^2}{2MR^2}\left(im - i\frac{\mathbf{\Phi}}{\phi_0}\right)^2 e^{im\phi} \\ &= \frac{\hbar^2}{2MR^2}\left(m - \frac{\mathbf{\Phi}}{\phi_0}\right)^2 \langle\phi|m\rangle\end{aligned}$$

Which demonstrates that the angular momentum eigenstates are energy eigenstates with

$$E_m = \frac{\hbar^2}{2MR^2}\left(m - \frac{\mathbf{\Phi}}{\phi_0}\right)^2$$

As a function of m the energy is a parabola.

(2)

With no loss of generality we can write:

$$\frac{\mathbf{\Phi}}{\phi_0} = m_0 + a$$

where $1/2 < a \leq 1/2$. Clearly $m_0 = round\left(\frac{\mathbf{\Phi}}{\phi_0}\right)$ is the integer that is closer to $\frac{\mathbf{\Phi}}{\phi_0}$. With this definition the ground state energy is for some $m = m_0$ and is equal to $E_G = \frac{\hbar^2}{2MR^2}a^2$ which is a periodic function of $\frac{\mathbf{\Phi}}{\phi_0}$ with period one. There are two extreme cases: $a = 0$ which means that $\frac{\mathbf{\Phi}}{\phi_0}$ is an integer. In this case there is a unique ground state with $E_G = 0$ and all the other states are twofold degenerate (the states with

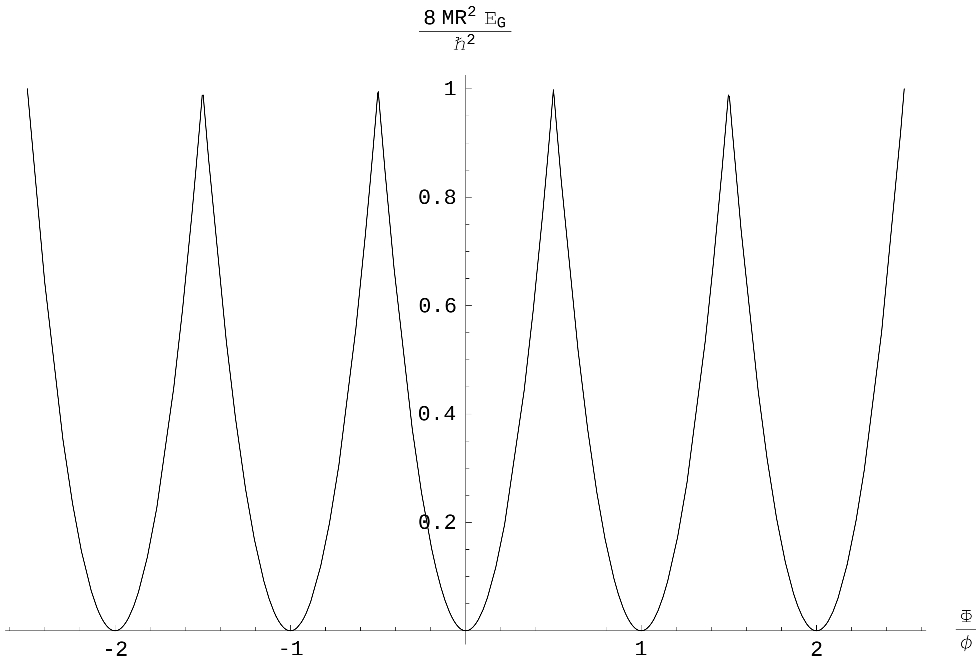

Figure 1: The ground state energy as a function of $\frac{\mathbf{\Phi}}{\phi_0}$. At the tips where $\frac{\mathbf{\Phi}}{\phi_0} = \frac{1}{2} + m$ the ground state is twofold degenerate and mixes the states $|m\rangle$ and $|m-1\rangle$ with energy $\frac{\hbar^2}{8MR^2}$. Elsewhere the ground state $|m\rangle$, where m is the closest integer to $\frac{\mathbf{\Phi}}{\phi_0}$, is not degenerate.

$m = q + \frac{\mathbf{\Phi}}{\phi_0}$ and $m = -q + \frac{\mathbf{\Phi}}{\phi_0}$ have the same energy). Another extreme case in the one in which $a = 1/2$ or that $\frac{\mathbf{\Phi}}{\phi_0}$ is an odd number. Then this number is at the same distance from two consecutive integers m_0 and $m_0 + 1$. The ground state is twofold degenerate in this case because:

$$E_{m_0+1} = \frac{\hbar^2}{2MR^2}\left(m_0 + 1 - m_0 - \frac{1}{2}\right)^2 = E_{m_0} = \frac{\hbar^2}{8MR^2}$$

Everywhere else the energy is going to fluctuate between 0 and $\frac{\hbar^2}{8MR^2}$ with period 1. At each period the energy is described by a parabola centered at some integer as shown in figure. At the tips there is twofold degeneracy.

(3)

The $|m\rangle$ are eigenstates of the current with eigenvalues:

$$j_\phi^{(m)} = -\frac{\phi_0}{2\pi MR}\left(m - \frac{\mathbf{\Phi}}{\phi_0}\right)$$

For the ground state $\frac{\mathbf{\Phi}}{\phi_0} = m_0 + a$ and $m = m_0$ in which case:

$$j_\phi^G = \frac{\phi_0}{2\pi MR}a$$

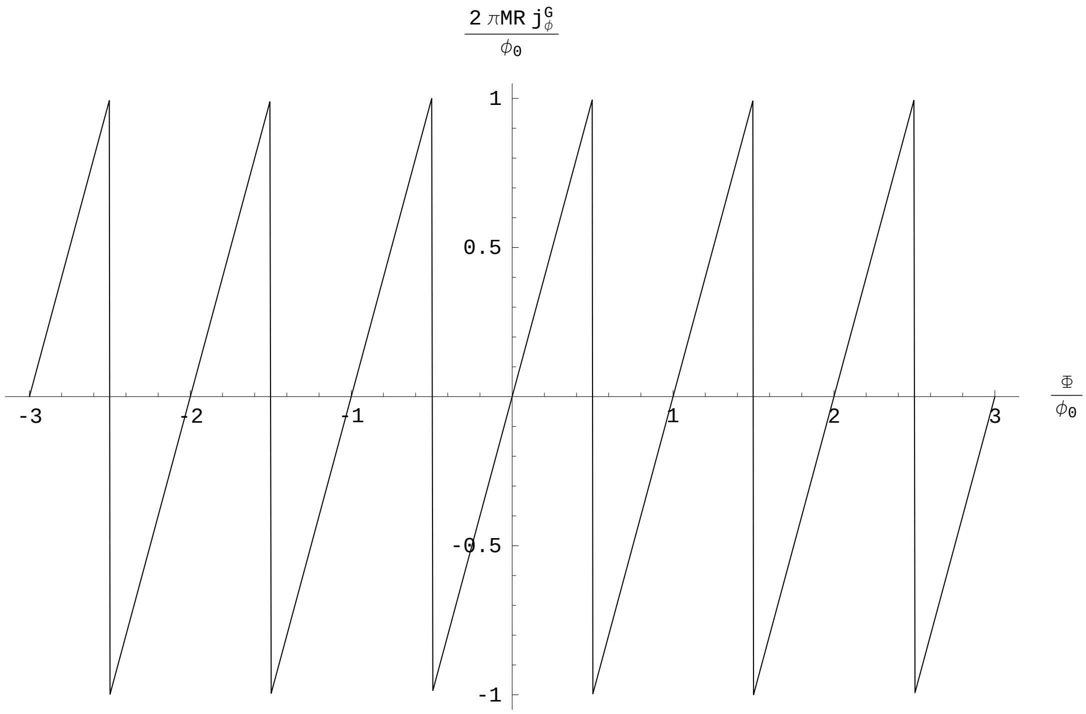

Figure 2: The ground state current as a function of $\frac{\mathbf{\Phi}}{\phi_0}$. The discontinuity is because of the ground state reconstruction.

The current will be a period function of $\frac{\mathbf{\Phi}}{\phi_0}$ with period 1. For $a = 0$ the current is zero and for $a = 1/2$ it gets its maximum value of $\frac{\phi_0}{4\pi MR}$.

(4)

Lets set $\frac{\mathbf{\Phi}}{\phi_0} = 1/2 + \delta$ where $\delta \ll 1$. There are two (almost) degenerate states that we will denote $|m = 0\rangle$ and $|m = 1\rangle$ respectively with energy $\frac{\hbar^2}{2MR^2}\left(\frac{\mathbf{\Phi}}{\phi_0}\right)^2$ and $\frac{\hbar^2}{2MR^2}\left(1 - \frac{\mathbf{\Phi}}{\phi_0}\right)^2$ respectively.

The $\hat{H}_0$ part of the Hamiltonian that does not contain the electric field is of course diagonal:

$$\begin{aligned}
\hat{H}_0 &= \frac{\hbar^2}{2MR^2}\begin{pmatrix} \left(1 - \frac{\mathbf{\Phi}}{\phi_0}\right)^2 & 0 \\ 0 & \left(\frac{\mathbf{\Phi}}{\phi_0}\right)^2 \end{pmatrix} \\
&= \frac{\hbar^2}{2MR^2}\frac{1}{2}\left[\left(\frac{\mathbf{\Phi}}{\phi_0}\right)^2 + \left(1 - \frac{\mathbf{\Phi}}{\phi_0}\right)^2\right]\mathbf{1} + \frac{\hbar^2}{2MR^2}\frac{1}{2}\left[\left(1 - \frac{\mathbf{\Phi}}{\phi_0}\right)^2 - \left(\frac{\mathbf{\Phi}}{\phi_0}\right)^2\right]\sigma_z
\end{aligned}$$

Where the basis for the matrix is $(|m = 1\rangle, |m = 0\rangle)$ (in that order).

The part of the hamiltonian $\hat{H}_1$ that depends on the electric field can be written as:

$$\hat{H}_1 = -eR\frac{1}{2}(E_x - iE_y)e^{i\phi} - eR\frac{1}{2}(E_x + iE_y)e^{-i\phi}$$

Clearly the $e^{i\phi}$ connects the $m = 0$ with the $m = 1$ state and the $e^{-i\phi}$ the $m = 1$ with the $m = 0$ state. Therefore:

$$\hat{H}_1 = -eR\frac{1}{2}\begin{pmatrix} 0 & E_x - iE_y \\ E_x + iE_y & 0 \end{pmatrix} = -eR\frac{1}{2}(E_x\sigma_x + E_y\sigma_y)$$

The four coefficients have units of energy and are:

$$\begin{aligned}
a_0 &= \frac{\hbar^2}{2MR^2}\frac{1}{2}\left[\left(\frac{\mathbf{\Phi}}{\phi_0}\right)^2 + \left(1 - \frac{\mathbf{\Phi}}{\phi_0}\right)^2\right] \approx \frac{\hbar^2}{8MR^2} \\
a_1 &= \frac{\hbar^2}{2MR^2}\frac{1}{2}\left[-\left(\frac{\mathbf{\Phi}}{\phi_0}\right)^2 + \left(1 - \frac{\mathbf{\Phi}}{\phi_0}\right)^2\right] \approx -\frac{\hbar^2}{2MR^2}\delta \\
a_2 &= \frac{1}{2}eRE_x \\
a_3 &= \frac{1}{2}eRE_y
\end{aligned}$$

Note to the students: if your expressions are different and you did not make a mistake it is because you swapped the $m = 0$ and $m = 1$ basis states.

(5)

The a_0 constant is nothing but a universal shift of the energy and it will be ignored (**1** commutes with anything anyway). The other three terms can be written as:

$$\hat{H} = \vec{B} \cdot \hat{\vec{S}}$$

where $\hat{\vec{S}} = \hbar\left(\sigma_x, \sigma_y, \sigma_z\right)/2$ is the effective spin and $\vec{B}$ plays the role of a (rescaled) magnetic field measured in units of frequency with components:

$$\begin{aligned}
B_x &= \frac{eR}{\hbar}E_x \\
B_y &= \frac{eR}{\hbar}E_y \\
B_z &= \frac{\hbar}{MR^2}\left(\frac{1}{2} - \frac{\mathbf{\Phi}}{\phi_0}\right)
\end{aligned}$$

In the spin-picture the spin up state corresponds to the $|m = 1\rangle$ and the spin down to the $|m = 0\rangle$ so that it is trivial to switch between the two pictures.

The equation of motion of $\hat{\vec{S}}$ is the same as in Problem 1 (if we ignore some coefficients):

$$i\hbar\frac{d}{dt}\hat{S}_i = \left[\hat{S}_i, H\right] = \sum_j B_j\left[\hat{S}_i, \hat{S}_j\right] = \sum_j B_j i\hbar\epsilon_{ijk}\hat{S}_k$$

or

$$\frac{d}{dt}\hat{\vec{S}} = \vec{B} \times \hat{\vec{S}}$$

This equation defines the counter clockwise precession of the polarization $\hat{\vec{S}}$ around the axis defined by the effective magnetic field. The precession frequency is just the magnitude of the effective magnetic field:

$$\omega = \left|\vec{B}\right| = \sqrt{\frac{e^2R^2}{\hbar^2}\left(E_x^2 + E_y^2\right) + \frac{\hbar^2}{M^2R^4}\delta^2}$$

To demonstrate the precession even further one can follow the analysis of the lecture notes (Spin, page 24).

(6)

Now the time dependent effective Hamiltonian is:

$$H = -\frac{\hbar^2}{2MR^2}\left(\frac{\partial}{\partial\phi} - i\frac{\mathbf{\Phi}}{\phi_0}\right)^2 - eRE_x\cos\phi\cos\omega t$$

We can carry out the same procedure to get a Hamiltonian $\hat{H} = \vec{B}(t)\cdot\hat{\vec{S}}$ where the fluctuating magnetic field is:

$$\begin{aligned} B_x &= \frac{eR}{\hbar}E_x\cos\omega t \\ B_y &= 0 \\ B_z &= \frac{\hbar}{MR^2}\delta \end{aligned}$$

where to simplify notation I set $\delta = \left(\frac{1}{2} - \frac{\mathbf{\Phi}}{\phi_0}\right)$.

The boundary condition of the problem is that at $t = 0$ the system is in the spin down effective state (or $m = 0$ state in the original picture) and we are asked to evaluate the probability that it appears in the spin up state after some time T. This spin flips probability is give by:

$$P(T) = \left|\left\langle\uparrow|\hat{U}(T,0)|\downarrow\right\rangle\right|^2$$

It is important to point out that if $\hbar\omega$ is comparable to the energy difference between the two states $m = 0$ and $m = 1$ and other excited states, which is is of the order $\frac{\hbar^2}{MR^2}$, the electric field will cause excitations and the simple two-state picture is no longer valid. We will have to assume that:

$$\omega \ll \frac{\hbar^2}{MR^2}$$

Although this problem looks very similar to the resonance problem.

In matrix form the Hamiltonian is:

$$\hat{H} = \begin{pmatrix} \frac{\hbar}{MR^2}\delta & \frac{eR}{\hbar}E_x\cos\omega t \\ \frac{eR}{\hbar}E_x\cos\omega t & \frac{\hbar}{MR^2}\delta \end{pmatrix}$$

Lets go to the rotated frame as we did in Problem 1:

$$|\psi_\omega(t)\rangle = e^{-i\omega t\hat{\sigma}_z/2}\,|\psi(t)\rangle$$

The equation of motion is again:

$$i\hbar\frac{d}{dt}\,|\psi_\omega(t)\rangle \quad = \quad (\hat{H}_\omega(t)+\omega\hat{S}_z)\,|\psi_\omega(t)\rangle$$

The rotated hamiltonian is:

$$\hat{H}_\omega(t)+\omega\hat{S}_z = \frac{\hbar}{2}\begin{pmatrix} \frac{\hbar}{MR^2}\delta & \frac{eR}{\hbar}E_x e^{-i\omega t}\cos\omega t \\ \frac{eR}{\hbar}E_x e^{i\omega t}\cos\omega t & -\frac{\hbar}{MR^2}\delta \end{pmatrix}$$

We will ignore the time dependent exponentials in this expression to get a time independent hamiltonian:

$$\hat{H}_\omega = \frac{\hbar}{2}\begin{pmatrix} \frac{\hbar}{MR^2}\delta & \frac{eR}{\hbar}E_x \\ \frac{eR}{\hbar}E_x & -\frac{\hbar}{MR^2}\delta \end{pmatrix} = \frac{\hbar}{2}\begin{pmatrix} B_z & B_x \\ B_x & -B_z \end{pmatrix}$$

This is the same problem as the one we solved in problem 1, but with different parameters. More specifically if we set $\omega_0 = -B_z$, $B_0 = B_z$, $B_1 = B_x$ and $\Omega = \sqrt{(B_z+\omega)^2+B_x^2}$ we can connect the two problems. In part 6 we evaluated the probability of a spin flip and found it equal to $\frac{\omega_0^2}{\Omega^2}\frac{B_0^2}{B_1^2}\sin^2\frac{\Omega t}{2} \rightarrow \frac{1}{\Omega^2}\frac{B_z^2}{B_x^2}\sin^2\frac{\Omega t}{2}$. Therefore the probability is a periodic function of time with:

$$T_0 = 2\pi/\Omega = \frac{2\pi}{\sqrt{\left(\frac{\hbar}{MR^2}\delta+\omega\right)^2+\left(\frac{eR}{\hbar}E_x\right)^2}} \approx \frac{2\pi}{\sqrt{\omega^2+\left(\frac{eR}{\hbar}E_x\right)^2}}$$

3 Harmonic Oscillators and Angular Momentum

As a clarifying note we have to mention that $\sigma_i^{\alpha\beta}$ is the element (α,β) (row α, column β) of the i^{th} Pauli matrix. Also the creation/annihilation operators satisfy the commutation relations:

$$\begin{aligned} \left[\hat{a}_i^\dagger,\hat{a}_j^\dagger\right] &= \left[\hat{a}_i^\dagger,\hat{a}_j^\dagger\right] = 0 \\ \left[\hat{a}_i,\hat{a}_j^\dagger\right] &= \delta_{ij} \end{aligned}$$

(1)

Lets evaluate the commutator:

$$\begin{aligned} \left[\hat{J}_i,\hat{J}_j\right] &= \frac{\hbar^2}{4}\sum_{\alpha,\beta}\sum_{\alpha',\beta'}\left[\hat{a}_\alpha^\dagger\hat{a}_\beta,\hat{a}_{\alpha'}^\dagger\hat{a}_{\beta'}\right]\sigma_i^{\alpha\beta}\sigma_j^{\alpha'\beta'} \\ &= \frac{\hbar^2}{4}\sum_{\alpha,\beta}\sum_{\alpha',\beta'}\left(\hat{a}_\alpha^\dagger\left[\hat{a}_\beta,\hat{a}_{\alpha'}^\dagger\right]\hat{a}_{\beta'}+\hat{a}_{\alpha'}^\dagger\left[\hat{a}_\alpha^\dagger,\hat{a}_{\beta'}\right]\hat{a}_\beta\right)\sigma_i^{\alpha\beta}\sigma_j^{\alpha'\beta'} \end{aligned}$$

$$
\begin{aligned}
&= \frac{\hbar^2}{4}\sum_{\alpha,\beta}\sum_{\alpha',\beta'}\left(\delta_{\beta\alpha'}\hat{a}^\dagger_\alpha\hat{a}_{\beta'}-\delta_{\alpha\beta'}\hat{a}^\dagger_{\alpha'}\hat{a}_\beta\right)\sigma_i^{\alpha\beta}\sigma_j^{\alpha'\beta'}\\
&= \frac{\hbar^2}{4}\left(\sum_{\alpha,\beta}\sum_{\beta'}\hat{a}^\dagger_\alpha\hat{a}_{\beta'}\sigma_i^{\alpha\beta}\sigma_j^{\beta\beta'}-\sum_{\alpha,\beta}\sum_{\alpha'}\delta_{\alpha\beta'}\hat{a}^\dagger_{\alpha'}\hat{a}_\beta\sigma_i^{\alpha\beta}\sigma_j^{\alpha'\alpha}\right)\\
&= \frac{\hbar^2}{4}\sum_{\alpha,\beta}\left(\sum_{\alpha}\sum_{\beta'}\left[\sigma_i\sigma_j\right]_{\alpha\beta'}\hat{a}^\dagger_\alpha\hat{a}_{\beta'}-\sum_{\beta}\sum_{\alpha'}\left[\sigma_j\sigma_i\right]_{\alpha'\beta}\right)\\
&= \frac{\hbar^2}{4}\sum_{\alpha,\beta}\left[\sigma_i\sigma_j-\sigma_j\sigma_i\right]_{\alpha\beta}\hat{a}^\dagger_\alpha\hat{a}_\beta\\
&= \frac{\hbar^2}{4}\sum_{\alpha,\beta}2\epsilon_{ijk}\sigma_k^{\alpha\beta}\hat{a}^\dagger_\alpha\hat{a}_\beta=\\
&= \hbar\epsilon_{ijk}\hat{J}_k
\end{aligned}
$$

Indeed the three J_i follow the angular momentum algebra.

(2)

It is instructive to write down explicitly all the member of the algebra:

$$
\begin{aligned}
\hat{J}_x &= \frac{\hbar}{2}\left(\hat{a}^\dagger_1\hat{a}_2+\hat{a}^\dagger_2\hat{a}_1\right)\\
\hat{J}_y &= \frac{\hbar}{2}\left(-i\hat{a}^\dagger_1\hat{a}_2+i\hat{a}^\dagger_2\hat{a}_1\right)\\
\hat{J}_z &= \frac{\hbar}{2}\left(\hat{N}_1-\hat{N}_2\right)\\
\hat{J}_+ &= \hat{J}_x+i\hat{J}_y=\hbar\hat{a}^\dagger_1\hat{a}_2\\
\hat{J}_- &= \hat{J}_x-i\hat{J}_y=\hbar\hat{a}^\dagger_2\hat{a}_1
\end{aligned}
$$

where $\hat{N}_i=\hat{a}^\dagger_i\hat{a}_i$. Note that J_z gives us the quantum number m.
We will use the identity:

$$
\begin{aligned}
\hat{\vec{J}}^2 &= \hat{J}_z^2+\frac{1}{2}\left(\hat{J}_+\hat{J}_-+\hat{J}_-\hat{J}_+\right)\\
&= \hbar^2\left[\frac{1}{4}\left(\hat{N}_1-\hat{N}_2\right)^2+\frac{1}{2}\hat{N}_1\left(1+\hat{N}_2\right)+\frac{1}{2}\hat{N}_2\left(1+\hat{N}_1\right)\right]\\
&= \hbar^2\left[\frac{1}{4}\left(\hat{N}_1+\hat{N}_2\right)^2+\frac{1}{2}\hat{N}_1+\frac{1}{2}\hat{N}_2\right]\\
&= \hbar^2\frac{\hat{N}}{2}\left(\frac{\hat{N}}{2}+1\right)
\end{aligned}
$$

where $\hat{N}=\hat{N}_1+\hat{N}_2$ is the total number operator. We see that the states with particular occupation numbers N_i correspond angular momentum eigenstates. The angular momentum quantum number is $j=\frac{N}{2}$.

Also since $\hat{J}_z = \frac{\hbar}{2}\left(\hat{N}_1 - \hat{N}_2\right) = \hbar m$ the $\hat{J}_z$ quantum number is $m = \frac{N_1 - N_2}{2}$:

$$\begin{aligned} j &= \frac{N_1 + N_2}{2} \\ m &= \frac{N_1 - N_2}{2} \end{aligned}$$

Since $N_i \geq 0$ then $j \geq 0$ as it should. Also for fixed j we have the constraint that $N_i \leq 2j$. Also $m = j - N_2$. We can vary $0 \leq N_2 \leq 2j$ at integer steps to get $-j \leq m \leq j$.

(3)

We want to fill the two oscillators with $N_1 = j+m$ and $N_2 = j-m$ particles. Let $|N_1, N_2\rangle_O = |j+m, j-m\rangle_O$ be the $|j,m\rangle$ eigenstate (The index O distinguishes between the angular momentum and the oscillator eigenstates). Lets apply the raising operator $\hat{J}_+ = \hbar\hat{a}_1^\dagger\hat{a}_2$on this state:

$$\begin{aligned} \hat{J}_+ |j,m\rangle &= \hbar\hat{a}_1^\dagger\hat{a}_2 |j+m, j-m\rangle_O \\ &= \hbar\sqrt{(j+m+1)(j-m)}\, |j+m+1, j-m-1\rangle_O \\ &= \hbar\sqrt{j(j+1) - m(m+1)}\, |j, m+1\rangle \end{aligned}$$

In the same transparent way we can derive the equation for the lowering operator $\hat{J}_- = \hbar\hat{a}_2^\dagger\hat{a}_1$:

$$\begin{aligned} \hat{J}_- |j,m\rangle &= \hbar\hat{a}_2^\dagger\hat{a}_1 |j+m, j-m\rangle_O \\ &= \hbar\sqrt{(j+m)(j-m+1)}\, |j+m+1, j-m-1\rangle_O \\ &= \hbar\sqrt{j(j+1) - m(m-1)}\, |j, m-1\rangle \end{aligned}$$

(4)

The matrix element of $\hat{K}^\dagger = \hat{a}_1^\dagger\hat{a}_2^\dagger$ and $\hat{K} = \hat{a}_2\hat{a}_1$ can be found after we apply them on particular states $|j,m\rangle = |j+m, j-m\rangle_O$:

$$\begin{aligned} \hat{K}^\dagger |j,m\rangle &= \sqrt{(j+m+1)(j-m+1)}\, |j+m+1, j-m+1\rangle_O = \sqrt{(j+1)^2 - m^2}\, |j+1, m\rangle \\ \hat{K} |j,m\rangle &= \sqrt{(j+m)(j-m)}\, |j+m-1, j-m-1\rangle_O = \sqrt{j^2 - m^2}\, |j-1, m\rangle \end{aligned}$$

As we see $\hat{K}^\dagger$ increases the total angular momentum j by one while keeping m constant. The adjoint $\hat{K}$ decreases the angular momentum by one and keeps m constant. We can express this as matrix elements:

$$\begin{aligned} \langle j', m'| \hat{K}^\dagger |j,m\rangle &= \delta_{m,m'}\delta_{j',j+1}\sqrt{(j+1)^2 - m^2} \\ \langle j', m'| \hat{K} |j,m\rangle &= \delta_{m,m'}\delta_{j',j-1}\sqrt{j^2 - m^2} \end{aligned}$$

Addition of spin angular momenta

(1)

There are two states for each spin and $2 \times 2 = 4$ states in the product basis. The product basis has states $|m_1\rangle \otimes |m_2\rangle$ where each m_i takes values $\pm 1/2$. A common way to denote these 4 states is:

$$|\uparrow\uparrow\rangle, |\uparrow\downarrow\rangle, |\downarrow\uparrow\rangle, |\downarrow\downarrow\rangle$$

(2)

The total angular momentum will take values from$|1/2 - 1/2| = 0$ to $|1/2 + 1/2| = 1$ therefore there are only two possibilities: $j = 0$ (singlet) or $j = 1$ (triplet). Formally:

$$\frac{1}{2} \otimes \frac{1}{2} = 1 \oplus 0$$

(3)

We start from $|\uparrow\uparrow\rangle = |j = 1, m = 1\rangle$ which has $m = 1$ and therefore corresponds to $j = 1$. Similarly the $|\downarrow\downarrow\rangle$ state corresponds to $j = 1$ with $m = -1$. To get the $m = 0$ triplet state we will apply the $\hat{J}_- = \hat{J}_-^{(1)} + \hat{J}_-^{(2)}$:

$$\begin{aligned}
\hat{J}_- |j = 1, m = 1\rangle &= \hat{J}_-^{(1)} \left|\frac{1}{2}\right\rangle \otimes \left|\frac{1}{2}\right\rangle + \left|\frac{1}{2}\right\rangle \otimes \hat{J}_-^{(2)} \left|\frac{1}{2}\right\rangle \Rightarrow \\
\sqrt{1(1+1) - 1(1-1)}\, |j = 1, m = 0\rangle &= \sqrt{\frac{1}{2}\left(\frac{1}{2}+1\right) - \frac{1}{2}\left(\frac{1}{2}-1\right)} \left(\left|-\frac{1}{2}\right\rangle \otimes \left|\frac{1}{2}\right\rangle + \left|\frac{1}{2}\right\rangle \otimes \left|-\frac{1}{2}\right\rangle\right) \Rightarrow \\
\sqrt{2}\, |j = 1, m = 0\rangle &= \left|-\frac{1}{2}\right\rangle \otimes \left|\frac{1}{2}\right\rangle + \left|\frac{1}{2}\right\rangle \otimes \left|-\frac{1}{2}\right\rangle
\end{aligned}$$

We can summarize this as:

$$|j = 1, m = 0\rangle = \frac{|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle}{\sqrt{2}}$$

Now we have constructed the whole $j = 1$ space. The $j = 0$ space contains exactly one state which is some linear combination of $m = 0$ states:

$$|j = 0, m = 0\rangle = a \left|-\frac{1}{2}\right\rangle \otimes \left|\frac{1}{2}\right\rangle + b \left|\frac{1}{2}\right\rangle \otimes \left|-\frac{1}{2}\right\rangle$$

This state must be orthogonal to the $|j = 1, m = 0\rangle$:

$$0 = \langle j = 1, m = 0 | j = 0, m = 0\rangle = \frac{a + b}{\sqrt{2}}$$

Therefore $a = -b = \frac{1}{\sqrt{2}}$ where we made use of the normalization condition. To summarize the triplet states are:

$$\begin{aligned}
|j = 1, m = 1\rangle &= |\uparrow\uparrow\rangle \\
|j = 1, m = 0\rangle &= \frac{|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle}{\sqrt{2}} \\
|j = 1, m = -1\rangle &= |\downarrow\downarrow\rangle \\
|j = 0, m = 0\rangle &= \frac{|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle}{\sqrt{2}}
\end{aligned}$$

and you will see these equations a ~billion times before the age of 25.