Physics 581: Quantum Mechanics II
Department of Physics, UIUC
Spring Semester 2007
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Problem Set No. 2:
Path Integrals and Quantum Mechanics
Due Date: 2/19/2007
No late sets will be accepted

1 Path Integral for a charged particle moving on a plane in the presence of a perpendicular magnetic field.

Consider a particle of mass m and charge e moving on a plane in the presence of an external uniform magnetic field perpendicular to the plane and with strength B. Let $\vec{r} = (x_1, x_2)$ and $\vec{p} = (p_1, p_2)$ represent the components of the coordinate $\vec{r}$ and of the momentum $\vec{p}$ of the particle. The Lagrangian for the particle is

$$L = \frac{1}{2} m \left(\frac{d\vec{r}}{dt} \right)^2 + \frac{e}{c} \frac{d\vec{r}}{dt} \cdot \vec{A}(\vec{r}) \tag{1}$$

1. Find the relation between the momentum $\vec{p}$ and the coordinate $\vec{r}$ and explain how is the momentum related with the velocity $\vec{v} = \frac{d\vec{r}}{dt}$ in this case.

2. Show that the classical Hamiltonian of for this problem is

$$H(q,p) = \frac{1}{2m} (\vec{p}^{\,2} - \frac{e}{c} \vec{A}(\vec{r}))^2 \tag{2}$$

   where $\vec{A}(\vec{r})$ is the vector potential for a uniform magnetic field, normal to the plane, and of magnitude B. In what follows, we will always write the vector potential in the gauge $\vec{\bigtriangledown} \cdot \vec{A}(\vec{r}) = 0$, where it is given by

$$A_1(\vec{r}) = -\frac{B}{2} x_2 \qquad A_2(\vec{r}) = \frac{B}{2} x_1 \tag{3}$$

3. Use canonical quantization to find the quantum mechanical Hamiltonian and the commutation relations for the observables.

4. Derive the form of the *path integral*, as a sum over the histories of the position $\vec{r}(t)$ of the particle, for the transition amplitude of the process in

which the particle returns to its initial location $\vec{r}_0$ at time t_f having left that point at t_i *i. e.*

$$\langle \vec{r}_0, t_f | \vec{r}_0, t_i \rangle \tag{4}$$

where $\vec{r}_0$ is an arbitrary point of the plain and $|t_f - t_i| \to \infty$. What is the form of the action? What initial and final conditions should be satisfied by the histories $\vec{r}(t)$?

5. Consider now the "ultra-quantum" limit $m \to 0$. Find the form of the action S in this limit for a path which begins and ends at $\vec{r}_0$. Find a geometric interpretation for this formula.
(Hint: at some point you may have to use Stokes theorem). Are there any ambiguities involved in the evaluation of this formula?. Think of the regions enclosed by the path and left outside of the path. What condition should satisfy the field strength B, for a plane of dimensions $L \times L$ (with $L \to \infty$), so that the *amplitude* $e^{\frac{i}{\hbar}S}$ is free from any ambiguities?.

2 Path Integral for the three-dimensional harmonic oscillator

Consider a harmonic oscillator of mass m and frequency ω in *three* dimensions. We will denote the position vector of the oscillator by $\vec{r} = (x, y, z)$. The classical Hamiltonian is

$$H(\vec{r}, \vec{p}) = \frac{\vec{p}^{\,2}}{2m} + \frac{1}{2} m\omega^2 \vec{r}^{\,2} \tag{5}$$

Derive an expression for the path integral for the matrix element

$$\langle \vec{r}_f = 0, t_f | \vec{r}_i = 0, t_i \rangle \tag{6}$$

for this three-dimensional oscillator, where $t_f \to +\infty$ and $t_i \to -\infty$. Make sure that you explain how this limit is taken.
Hint: you will find it convenient to write the path integral in terms of the histories of the three components $x(t)$, $y(t)$, and $z(t)$.

3 Transitions in the forced one-dimensional oscillator

Consider a one-dimensional oscillator of mass m and frequency ω, labeled by the coordinate $q(t)$ on an infinite line. The oscillator is subject to an external force $J(t)$ of the form

$$J(t) = W \frac{\tau}{t^2 + \tau^2} \tag{7}$$

1. What are the units of W? Use W and m to construct a quantity with units of energy.

2. Use the formulas derived in class for the path integral for the forced harmonic oscillator to calculate the amplitude

$$\langle q = 0, t_f | q = 0, t_i \rangle \tag{8}$$

   for $t_i \to -\infty$ and $t_f \to +\infty$.

3. How does the expression you found depend on W, τ, m and ω? Give a physical interpretation to this dependence by looking at the extreme regimes of τ large and small (relative to what?).

4. What dependence on W would you have expected to find in the Born approximation? And in higher orders in perturbation theory?