

1 Charged Particle on a Plane

1)

The canonical momentum:

$$\vec{p} = \vec{\nabla}_r L = m\dot{\vec{r}} + \frac{e}{c}\vec{A}(\vec{r})$$

2)

$$\begin{aligned} H &= \dot{\vec{r}} \cdot \vec{p} - \frac{1}{2}m\dot{\vec{r}}^2 - \frac{e}{c}\dot{\vec{r}} \cdot \vec{A}(\vec{r}) \\ &= \dot{\vec{r}} \cdot \left(\vec{p} - \frac{e}{c}\vec{A}(\vec{r}) \right) - \frac{1}{2}m\dot{\vec{r}}^2 \\ &= \frac{1}{2}m\dot{\vec{r}}^2 \\ &= \frac{1}{2m} \left(\vec{p} - \frac{e}{c}\vec{A}(\vec{r}) \right)^2 \end{aligned}$$

3)

We expand the Hamiltonian:

$$H = \frac{1}{2m}\vec{p}^2 + \frac{e^2}{2mc^2}\vec{A}^2(\vec{r}) - \frac{e}{mc}\vec{p} \cdot \vec{A}(\vec{r})$$

In the quantum mechanical Hamiltonian the only ambiguity is the term $\vec{p} \cdot \vec{A}(\vec{r}) \rightarrow \frac{1}{2}\hat{\vec{p}} \cdot \vec{A}(\vec{r}) + \frac{1}{2}\vec{A}(\vec{r}) \cdot \hat{\vec{p}}$. But because $\vec{\nabla} \cdot \vec{A}(\vec{r}) = 0$ $\hat{\vec{p}} \cdot \vec{A} = \vec{A} \cdot \hat{\vec{p}}$ (to understand that better assume that each side is applied in some arbitrary function $\psi(\vec{r})$). The quantum Hamiltonian is:

$$\begin{aligned} H &= \frac{1}{2m}\hat{\vec{p}}^2 + \frac{e^2}{2mc^2}\vec{A}^2(\vec{r}) - \frac{e}{mc}\vec{A}(\vec{r}) \cdot \hat{\vec{p}} \\ &= \frac{\hat{p}_1^2 + \hat{p}_2^2}{2m} + \frac{e^2 B^2}{8mc^2}(\hat{x}_1^2 + \hat{x}_2^2) + \frac{eB}{2mc}(\hat{x}_2\hat{p}_1 - \hat{x}_1\hat{p}_2) \end{aligned}$$

where $p_i = -i\hbar\partial_i$.

4)

We are now going to evaluate the path integral by summing over all histories that start and end at the same point. The boundary conditions are thus:

$$\vec{r}(t_i) = \vec{r}(t_f) = \vec{r}_0$$

The path integral is:

$$\langle \vec{r}_f, t_f | \vec{r}_i, t_i \rangle = \int D\vec{p}D\vec{r} e^{iS/\hbar}$$

where

$$S = \int_{t_i}^{t_f} dt \left(\vec{p} \cdot \dot{\vec{r}} - H(\vec{p}, \vec{r}) \right)$$

is the classical action for the problem. At first we integrate out the momentum \vec{p} which will give us an expression involving \vec{r} and $\dot{\vec{r}}$. The relevant integral is:

$$\begin{aligned} \int_{-\infty}^{\infty} \frac{dp_1 dp_2}{(2\pi\hbar)^2} e^{\frac{i}{\hbar}\epsilon\left(\vec{p}\cdot\dot{\vec{r}} - \frac{1}{2m}(\vec{p} - \frac{\epsilon}{c}\vec{A})^2\right)} &= \int_{-\infty}^{\infty} \frac{dp_1 dp_2}{(2\pi\hbar)^2} e^{\frac{i}{\hbar}\epsilon\left((\vec{p} + \frac{\epsilon}{c}\vec{A})\cdot\dot{\vec{r}} - \frac{1}{2m}\vec{p}^2\right)} \\ &= e^{\frac{i}{\hbar}\epsilon\frac{\epsilon}{c}\dot{\vec{r}}\cdot\vec{A}} \frac{m\hbar}{2\pi\epsilon} e^{\frac{i}{\hbar}m\dot{\vec{r}} + \frac{i}{\hbar}\epsilon\frac{\epsilon}{c}\dot{\vec{r}}\cdot\vec{A}} \end{aligned}$$

where ϵ is the time step. In the second equality we perform a shift $\vec{p} \rightarrow \vec{p} + \frac{\epsilon}{c}\vec{A}$ and in the third equality we just use Gauss' integral. The path integral then becomes:

$$\langle \vec{r}_f, t_f | r_i, t_i \rangle = \int D\vec{r} e^{\frac{i}{\hbar} \int_{t_i}^{t_f} dt L(\dot{\vec{r}})}$$

where $L(\dot{\vec{r}})$ is the classical Lagrangian.

5)

In the particular case $m = 0$ the Lagrangian in the exponent will become just $L = \frac{\epsilon}{c}\vec{A} \cdot \dot{\vec{r}}$ and the exponent will become:

$$\int_{t_i}^{t_f} dt L(\dot{\vec{r}}) = \int_{t_i}^{t_f} dt \frac{\epsilon}{c} \vec{A} \cdot \dot{\vec{r}} = \frac{\epsilon}{c} \oint \vec{A} \cdot d\vec{r}$$

The integral is along a closed path. The path integral will then become:

$$\langle \vec{r}_f, t_f | r_i, t_i \rangle = \int D\vec{r} e^{\frac{2\pi i}{\Phi_q} \oint \vec{A} \cdot d\vec{r}}$$

where the quantum of flux $\Phi_q = \frac{hc}{e}$ has been introduced. This integral in the exponent is gauge invariant because the endpoints of the path are the same.

According to Stoke's theorem, the quantity $\oint \vec{A} \cdot d\vec{r}$ is equal to the magnetic flux Φ going through the closed path if this is a positively oriented graph (the positive direction is commensurate to the direction of the magnetic field via the right hand rule). Also it is equal to minus the magnetic flux $-(BL^2 - \Phi) = \Phi - BL^2$ through the area outside the path, which is now viewed as negatively oriented. This means that the term BL^2 should not contribute which is possible if

$$e^{-2\pi i \frac{BL^2}{\Phi_q}} = 1 \Rightarrow BL^2 = n\Phi_q$$

That is in order for the ambiguity to be removed the total flux throught the plane has to be a integer multiple of the flux quantum.

2 Path Integral for the three-dimensional harmonic oscillator

The 3D harmonic oscillator is nothing but three independent 1D harmonic oscillators. The Hamiltonian can be broken into three parts:

$$\begin{aligned} H(\vec{p}, \vec{r}) &= H_1(p_1, x_1) + H_2(p_2, x_2) + H_3(p_3, x_3) \\ H_i(p_i, r_i) &= \frac{p_i^2}{2m} + \frac{1}{2}m\omega^2 x_i^2 \end{aligned}$$

The path integral can be factorized into three parts:

$$\begin{aligned} \langle \vec{r}_f = 0, t_f | \vec{r}_i, t_i \rangle &= \int D\vec{p} D\vec{r} e^{\frac{i}{\hbar} \int dt (\vec{p} \cdot \dot{\vec{r}} - H(\vec{p}, \vec{r}))} \\ &= \int Dp_1 Dx_1 e^{\frac{i}{\hbar} \int dt (p_1 \cdot \dot{x}_1 - H_1)} \times \\ &\quad \int Dp_2 Dx_2 e^{\frac{i}{\hbar} \int dt (p_2 \cdot \dot{x}_2 - H_2)} \times \\ &\quad \int Dp_3 Dx_3 e^{\frac{i}{\hbar} \int dt (p_3 \cdot \dot{x}_3 - H_3)} \end{aligned}$$

The three integrals are identical and so we can write:

$$\begin{aligned} \langle \vec{r}_f = 0, t_f | \vec{r}_i = 0, t_i \rangle &= \left[\int Dp Dx e^{\frac{i}{\hbar} \int dt (p \cdot \dot{x} - H)} \right]^3 \\ &= \left[\int Dx e^{\frac{i}{\hbar} \int dt \frac{1}{2} m (\dot{x}^2 - \omega^2 x^2 + i\eta x^2)} \right]^3 \end{aligned}$$

where in the last line we performed the momentum integration like in the lecture notes and we also added the term $i\eta x^2$ with $\eta > 0$ which guarantees convergence. This term will be discussed at the end. The 1D paths are taken with $x_i = x_f = 0$ and $Dx = \prod_{i=1}^{N-1} \sqrt{\frac{m\hbar}{2\pi\epsilon}} dx_i$. Also since nothing depends on time explicitly we can relabel the time coordinate so that $t_i = 0$ and $t_f = T$.

We write:

$$\int_{t_i}^{t_f} dt (\dot{x}^2 - \omega^2 x^2) = [x\dot{x}]_{t_i}^{t_f} + \int_{t_i}^{t_f} dt x (-\partial^2 - \omega^2) x$$

From the lecture notes we have established that:

$$\int Dy e^{\frac{i}{\hbar} \frac{1}{2} m \int dt (\dot{y}^2 - \omega^2 y^2)} = \int Dy e^{\frac{i}{\hbar} \frac{1}{2} m \int dt \dot{x}^2} \left(\frac{\det(-\partial^2 - \omega^2)}{\det(-\partial^2)} \right)^{-1/2}$$

where the integral of the right hand side represents the free particle propagator. In p226 of Shankar one can see that that free particle propagator can be evaluated by performing the integration explicitly and the result is:

$$\sqrt{\frac{m}{2\pi i \hbar T}}$$

The ratio of the determinants from the lecture notes is given by:

$$\frac{\det(-\partial^2 - \omega^2)}{\det(-\partial^2)} = \frac{\psi_0^{(1)}(T)}{\psi_0^{(2)}(T)}$$

where $\psi_0^{(i)}(0) = 0$ and $\partial\psi_0^{(i)}(0) = 1$ and $\psi_0^{(1)}$ is an eigenvector of the operator of the numerator whereas $\psi_0^{(2)}$ is an eigenvector of the denominator. Solving the trivial eigenvalue problems and imposing these boundary conditions gives:

$$\begin{aligned} \psi_0^{(1)}(t) &= \frac{1}{\omega} \sin \omega t \\ \psi_0^{(2)}(t) &= t \end{aligned}$$

And the ratio of the determinants is:

$$\frac{\det(-\partial^2 - \omega^2)}{\det(-\partial^2)} = \frac{\sin \omega t}{\omega t}$$

Finally the 1D propagators is:

$$\int Dy e^{\frac{i}{\hbar} \frac{1}{2} m \int dt (\dot{x}^2 - \omega^2 x^2)} = \sqrt{\frac{m\omega T}{2\pi i \hbar T \sin \omega T}}$$

and the 3D propagator:

$$\langle \vec{r}_f = 0, t_f | r_i = \vec{0}, t_i \rangle = \left(\frac{m\omega}{2\pi i \hbar \sin \omega T} \right)^{3/2}$$

where $T = t_f - t_i$. To take the proper limit at infinity we introduce imaginary time $T = -i\tau$ and set it to infinity:

$$\langle \vec{r}_f = 0, t_f | r_i = \vec{0}, t_i \rangle = \left(\frac{m\omega}{2\pi \hbar \sinh \omega \tau} \right)^{3/2} \approx \left(\frac{m\omega}{\pi \hbar} e^{-\omega \tau} \right)^{3/2} = \left(\frac{m\omega}{\pi \hbar} e^{-i\omega T} \right)^{3/2}$$

There is another much more transparent way to understand this limit. Remember the little term $i\eta x^2$ with $\eta > 0$ which was introduced for convergence. We performed the evaluation of the integral but we seemingly ignored this term. Actually this term is equivalent to replacing ω with some $\omega - i\delta$ where $\delta > 0$. Then we get $(\omega - i\delta)^2 \approx \omega^2 - 2i\delta\omega = \omega^2 - i\eta$ where $\eta = 2\delta\omega > 0$. Therefore the frequency of the harmonic oscillator must have a slight negative imaginary part for the path integral to converge. In this case $\sin \omega T = \frac{1}{2} (e^{i\omega T + \delta T} - e^{-i\omega T - \delta T})$. At the limit $T \rightarrow \infty$ the second term vanishes and we get the aforementioned result.

3 Transitions in the forced one-dimensional oscillator

1)

Since J has the units of force and τ the units of time, W has the units of force times time which is momentum. Then W^2/m has units of energy and measures the kinetic energy that the particle acquires from the external force.

2)

From page 87 of the lecture notes we get:

$$\begin{aligned} \lim_{t_f, i \rightarrow \pm\infty} \langle q = 0, t_f | q = 0, t_i \rangle_J &= \lim_{t_f, i \rightarrow \pm\infty} \langle q = 0, t_f | q = 0, t_i \rangle_{J=0} \times \\ &\exp \left(\frac{i}{2\hbar} \int_{-\infty}^{\infty} dt \int_{-\infty}^{\infty} dt' J(t) \frac{i}{2\omega m} e^{-i\omega|t-t'| - \epsilon|t-t'|} J(t') \right) \end{aligned}$$

where the $\epsilon > 0$ has been introduced for convergence. After setting $t/\tau \rightarrow t$ and $t'/\tau \rightarrow t'$ the exponent will read:

$$-\frac{W^2/m}{4\hbar\omega} \int_{-\infty}^{\infty} dt \int_{-\infty}^{\infty} dt' \frac{1}{t^2 + 1} e^{-z|t-t'|} \frac{1}{t'^2 + 1}$$

where $z = \epsilon + i\omega\tau$. To proceed we make the transformation $t = u + v$ and $t' = u - v$ so that $dt dt' = 2du dv$. The exponent will then become:

$$-\frac{W^2/m}{4\hbar\omega} 2 \int_{-\infty}^{\infty} dv e^{-2z|v|} \int_{-\infty}^{\infty} du \frac{1}{(u+v)^2 + 1} \frac{1}{(u-v)^2 + 1}$$

The inner integral converges uniformly and can be easily evaluated using residues to give $\frac{\pi/2}{1+v^2}$. The outer integral we break into two parts (with zero being the midpoint) and we get:

$$-\frac{W^2/m}{4\hbar\omega} 2\pi \int_0^\infty dv \frac{e^{-2zv}}{1+v^2} = -\frac{W^2/m}{4\hbar\omega} 2\pi \left(\int_0^\infty dv \frac{\cos(2\omega\tau v)}{1+v^2} + i \int_0^\infty dv \frac{\sin(2\omega\tau v)}{1+v^2} \right)$$

where we set $\epsilon = 0$ because the integral converges anyway. The imaginary part of this integral can be expressed with respect to special functions (Cosine integral function and Sine integral function) and it will only give a relative phase so we may as well ignore it. The real part can be calculated using residues:

$$\int_0^\infty dv \frac{\cos(2\omega\tau v)}{1+v^2} = \frac{1}{2} \int_{-\infty}^\infty dv \frac{\cos(2\omega\tau v)}{1+v^2} = \frac{1}{2} \Re \int_{-\infty}^\infty dv \frac{\exp(2i\omega\tau v)}{1+v^2} = \frac{\pi}{2} e^{-2\omega\tau}$$

Putting this all together the exponent will become:

$$G = -\frac{W^2/m}{4\hbar\omega} \pi^2 (e^{-2\omega\tau} + i\phi(\omega\tau))$$

Finally the amplitude of return for this forced Harmonic oscillator will be:

$$\langle q = 0, t_f | q = 0, t_i \rangle_J = \exp \left[-\frac{W^2/m}{4\hbar\omega} \pi^2 (e^{-2\omega\tau} + i\phi(\omega\tau)) \right] \left(\frac{m\omega}{\pi\hbar} e^{-i\omega T} \right)^{1/2}$$

3)

Note that the expression depends only on the dimensionless parameters $\omega\tau$ and the ratio of the energies $\frac{W^2/m}{4\hbar\omega}$. In the $\omega\tau \ll 1$ limit the external force is much faster than the natural period of the system ω^{-1} and the effect of the force is essentially that of an impulsive force $J(t) = \pi W \delta(t)$. In this limit the exponential in the integral $\int_0^\infty dv \frac{e^{-2zv}}{1+v^2}$ can be treated as a constant and the exponent will be $-\frac{W^2/m}{4\hbar\omega} \pi^2$. The probability of return depends on the ratio of the energies $\frac{W^2/m}{\hbar\omega}$. If it is too large the electron is kicked too hard and the chance of return is zero. If it is too low the probability is the same as without any external force.

In the opposite limit $\omega\tau \gg 1$ the external force is too slow compared to the natural period of the oscillator. The real part of the exponent will vanish exponentially. In this limit the harmonic oscillator is unperturbed and the probability of return then becomes equal to the one without external force regardless of the size of W .

4)

The perturbation is:

$$H_1(t) = qJ(t)$$

Consider the following algebra:

$$\langle q = 0, t_f | q = 0, t_i \rangle_J = \langle q = 0 | U(t_f, t_i) | q = 0 \rangle = \sum_{m,n} \langle q = 0 | m \rangle \langle m | U(t_f, t_i) | n \rangle \langle n | q = 0 \rangle$$

where U is the time evolution operator. Here I cite the results from problem 1 part 4 of Homework 1. There we saw that when a Hamiltonian is proportional to q , each time it acts on the system it raises or lowers the energy by one unit $\hbar\omega$. From this we deduce that if the order of the perturbation is odd the perturbation will connect states with energy differing by an odd number of $\hbar\omega$ and when it is even by an even number of $\hbar\omega$. So for odd order, one of the m and n in the summation will be odd and the other even and $\langle q = 0 | m \rangle \langle n | q = 0 \rangle = 0$ because all odd eigen states have a node at the center. Therefore all odd orders in perturbation, including the Born approximation, will vanish. This is consistent with the W^2 dependence that we found earlier.