

Physics 581: Quantum Mechanics II
Department of Physics, UIUC
Spring Semester 2007
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Problem Set No. 1:
Time-dependent Perturbation Theory
Due Date: 2/5/2007
No late sets will be accepted

1 Harmonic Oscillator

Consider a one-dimensional linear harmonic oscillator of mass m . The unperturbed Hamiltonian is H_0

$$H_0 = \frac{p^2}{2m} + \frac{1}{2}m\omega^2 x^2 \quad (1)$$

where x is the coordinate of the oscillator and p is the momentum.

Note: You may find it useful to write the hamiltonian in terms of raising and lowering operators a^\dagger and a ,

$$a = \sqrt{\frac{m\omega}{2\hbar}}x + \frac{ip}{\sqrt{2m\hbar\omega}} \quad a^\dagger = \sqrt{\frac{m\omega}{2\hbar}}x - \frac{ip}{\sqrt{2m\hbar\omega}} \quad (2)$$

with commutation relations $[a, a^\dagger] = 1$.

1. Write H_0 in terms of a and a^\dagger .
2. Compute the matrix elements $\langle n|a^\dagger|m\rangle$, where $|n\rangle$ and $|m\rangle$ are two eigenstates of H_0 .
3. Consider a perturbation $H_1(t)$ of the form

$$H_1(t) = gxe^{-t^2/\tau^2} \quad (3)$$

and assume that at some initial time $t_0 \ll -\tau$ the oscillator is in the eigenstate $|n\rangle$. Give a physical interpretation of this perturbation (assume that $\tau \rightarrow \infty$).

4. Use time-dependent perturbation theory to find the amplitude for finding the system in the state $|n+m\rangle$ due to the effects of H_1 to the *lowest non-vanishing order in perturbation theory*. To what order in perturbation theory is this amplitude *first* different from zero and why? Note: evaluate the expressions *only* at very long times, $t \gg \tau$ and at very short times $t \ll \tau$.

5. Consider now a perturbation of the form

$$H_2(t) = \gamma x^2 e^{-t^2/\tau^2} \quad (4)$$

and assume that at some initial time $t_0 \ll -\tau$ the oscillator is in the eigenstate $|n\rangle$. Give a physical interpretation of this perturbation (once again, you may assume that $\tau \rightarrow \infty$).

6. Use time-dependent perturbation theory to find the amplitude for finding the system in the state $|n+m\rangle$ due to the effects of H_2 to the *lowest non-vanishing order in perturbation theory*. To what order in perturbation theory is this amplitude *first* different from zero and why? Note: evaluate the expressions *only* at very long times, $t \gg \tau$ and at very short times $t \ll \tau$.

2 Ionization of the Hydrogen Atom

A uniform electric field

$$\vec{\mathcal{E}} = \mathcal{E} \cos \omega t \vec{e}_x \quad (5)$$

is applied to a *hydrogen* atom. Assume that the proton is heavy enough so that it remains static at all times. For the purposes of this problem you can ignore the electron spin and thus label the hydrogen bound states by $|n, l, m\rangle$ where, as usual, the quantum numbers n, l, m are integers satisfying $n = 1, \dots, \infty$, $0 \leq l \leq n - 1$ and $|m| \leq l$ and the energy levels are $E_{nlm} = -13.6 \text{ eV}/n^2$. We will assume that the perturbation is turned on adiabatically slowly in the remote past (you can accomplish that with a suitable adiabatic prefactor in front of the perturbation, $e^{\eta t}$, with $\eta \rightarrow 0^+$.)

1. Consider the $n = 2$ states of hydrogen in the presence of the perturbing electric field, *i. e.* at time $t_0 \rightarrow -\infty$ the hydrogen atom is in one of the four $|2, l, m\rangle$ states. Use time-dependent perturbation theory to find for which value of the principal quantum number n the transition $2 \rightarrow n$ is more likely (*i. e.* it has the *largest probability*). Calculate the contribution of this process to the lifetime of the $n = 2$ states of hydrogen. Note: be careful to consider all possible allowed initial and final states.
2. Calculate the *ionization rate* of hydrogen, initially in its ground state, as a function of ω , *i. e.* at time $t_0 \rightarrow -\infty$ the hydrogen atom is in its *ground state*, $|1, 0, 0\rangle$. For the sake of simplicity you may assume that the scattering states of hydrogen are well approximated by plane waves. Determine the angular distribution of the emitted electrons.

3 Instantaneous Perturbation

Consider a system which at times $t < 0$ is in some eigenstate $|i\rangle$ of a certain Hamiltonian H_0 . The system is subject to the action of an *instantaneous* perturbation $H_{\text{int}}(t) = \hat{V}\delta(t)$. Derive a formula for the transition amplitude, to

first order in perturbation theory, from state $|i\rangle$ for $t < 0$ to another state $|f\rangle$ at time $t > 0$.

4 Coulomb Excitation

A hydrogen atom is fixed at the origin of a coordinate system. A heavy particle of charge Ze , which we shall regard as classical, is sent towards the atom at speed v along the trajectory $\vec{R}(t) = vt\vec{e}_x + d\vec{e}_y$.

1. What is the form of the operator $H_1(t)$ (in the Schrödinger representation) which represents the Coulomb interaction between the *electron* of the hydrogen atom (which orbits the proton and has charge $-e$) and the heavy particle flying by. We will denote by \vec{r} the position vector of the *electron*. Write $H_1(t)$ in terms of the operator \vec{r} , Z , e and the position vector of the particle $\vec{R}(t)$.
2. Regard the interaction with the classical particle as a time-dependent perturbation. Derive a formula for the first transition probability $P_{n \rightarrow m}$ for a transition between the eigenstate $|n\rangle$ (with energy E_n) to an orthogonal eigenstate $|m\rangle$ (with energy E_m). [Do not do any integrals here!].
3. Assume that d^2 is large compared with mean square radius of the electron orbitals in both eigenstates $|n\rangle$ and $|m\rangle$ and show that, to leading order in perturbation theory,

$$P_{n \rightarrow m} \approx \frac{Z^2 e^4}{\hbar^2} \left| \int_{-\infty}^{+\infty} dt \frac{v \langle m|x|n\rangle + d \langle m|y|n\rangle}{(v^2 t^2 + d^2)^{\frac{3}{2}}} e^{i \frac{E_m - E_n}{\hbar} t} \right|^2 \quad (6)$$

where the operators x and y are components of the position vector of the electron \vec{r} .

4. What range of values of t give the dominant contribution to the integral of Eq. 6? Call this time scale τ .
5. You will now have to evaluate the integral of Eq. 6. Use the following integrals

$$\int_0^\infty dx \frac{x \sin ax}{(x^2 + \beta^2)^{3/2}} = a K_0(a\beta) \quad \int_0^\infty dx \frac{\cos ax}{(x^2 + \beta^2)^{3/2}} = -\frac{1}{\beta} \frac{\partial}{\partial \beta} K_0(a\beta) \quad (7)$$

where $K_0(z)$ is the Bessel function. Suppose now that $\frac{E_m - E_n}{\hbar} \tau \ll 1$. Find the probability $P_{n \rightarrow m}$ by evaluating the integral of Eq. 6 in this approximation. You will need to use the asymptotic expansion of the Bessel function

$$K_0(z) = \ln\left(\frac{2}{z}\right) + \dots \quad (8)$$

for $|z| \ll 1$.