1 Harmonic Oscillator

1)

$$H_0 = \hbar\omega\left(a^\dagger a + \frac{1}{2}\right)$$

2)

Remember that:

$$\begin{aligned} a\left|n\right\rangle &= \sqrt{n}\left|n-1\right\rangle \\ a^\dagger\left|n\right\rangle &= \sqrt{n+1}\left|n+1\right\rangle \end{aligned}$$

from witch we get:

$$\left\langle n\right| a^\dagger \left|m\right\rangle = \sqrt{m+1}\delta_{n,m+1}$$

And by iterating this:

$$\begin{aligned} \left\langle n+m\right| a^{\dagger m} \left|n\right\rangle &= \sqrt{\frac{(n+m)!}{n!}} \\ \left\langle n-m\right| a^{m} \left|n\right\rangle &= \sqrt{\frac{n!}{(n-m)!}} \end{aligned}$$

3)

It is the very slow switching on and off of an electric field. This electric field adds a linear term to the potential and shifts the location of its minimum. Therefore this is the adiabatic shift of the center of the Harmonic oscillator. We start with the system having some energy E_n. We apply the perturbation for a time of approximately τ. We switch off the perturbation and then measure the energy of the system which will in general not be in a clean state any more.

4)

In the interaction picture:

$$\begin{aligned} a(t) &= e^{-i\omega t}a \\ a^\dagger(t) &= e^{i\omega t}a^\dagger \end{aligned}$$

To obtain these relationships, start with the definition $a(t) = e^{iH_0t/\hbar}ae^{-iH_0t/\hbar}$, take the derivative and use the commutator: $[a, H_0] = \hbar\omega a$.

The perturbation will then become:

$$H_I(t) = \hbar g' e^{-t^2/\tau^2}\left(ae^{-i\omega t} + a^\dagger e^{i\omega t}\right)$$

where:

$$\hbar g' = g\sqrt{\frac{\hbar}{2m\omega}}$$

Therefore each time the system interacts with the perturbation its energy increases or decreases by $\hbar\omega$ with equal amplitude. In the minimum order the energy will either always increase ($m > 0$) in the intermediate steps or it will always decrease ($m < 0$). In the former case only the creation operator terms will contribute and in the latter only the annihilation. Therefore the first non vanishing perturbative order is m.

In the interaction picture the m^{th} perturbative correction of the time evolution operator is:

$$U_I^{(m)}(t) = \frac{(-i)^m}{\hbar^m}\int_{t_0}^{t} dt_1 \int_{t_0}^{t_1} dt_2 \cdots \int_{t_0}^{t_{m-1}} dt_m H_I(t_1)H_I(t_2)\cdots H_I(t_m)$$

If $m > 0$ the first contribution comes from the m^{th} order with all creation operators:

$$U_I^{(m)}(t) = (-ig'a^\dagger)^m \int_{t_0}^{t} dt_1 \int_{t_0}^{t_1} dt_2 \cdots \int_{t_0}^{t_{m-1}} dt_m e^{-\frac{1}{\tau^2}(t_1^2+t_2^2+\cdots+t_m^2)} e^{i\omega(t_1+t_2+\cdots+t_m)}$$

The integral gets greatly simplified if one notices that the integrand is symmetric under any permutation of the integration variables t_i so that:

$$U_I^{(m)}(t) = \frac{(-ig'a^\dagger)^m}{m!}\left[\int_{t_0}^{t} dt_1 e^{-t_1^2/\tau^2+i\omega t_1}\right]^m$$

Since $t_0 \ll \tau$ the lower limit in the integration can be taken to $-\infty$. The integral in the brackets can be evaluated as follows:

$$\begin{aligned}
\int_{t_0}^{t} dt_1 e^{-t_1^2/\tau^2+i\omega t_1} &= \tau\int_{-\infty}^{t/\tau} dx e^{-x^2+i\omega\tau x} \\
&= \tau e^{-\omega^2\tau^2/4}\int_{-\infty}^{t/\tau} dx e^{-(x-i\omega\tau/2)^2} \\
&= \tau e^{-\omega^2\tau^2/4}\int_{-\infty}^{t/\tau - i\omega\tau/2} dx e^{-x^2} \\
&= \Delta(\omega)\pi\left(1+erf(t/\tau - i\omega\tau/2)\right)
\end{aligned}$$

where $\Delta(\omega) = \frac{\tau}{2\sqrt{\pi}}e^{-\omega^2\tau^2/4}$ which evolves into a delta function as $\tau \to \infty$ and $erf(z) = \frac{2}{\sqrt{\pi}}\int_0^z e^{-t^2}dt$ is the error function.

At short enough times $t \ll \tau$:

$$\int_{-\infty}^{t} dt_1 e^{-t_1^2/\tau^2+i\omega t_1} = \Delta(\omega)\pi\left(1+erf(-i\omega\tau/2)\right)+O(t)$$

For long enough times $t \gg \tau$:

$$\int_{-\infty}^{t} dt_1 e^{-t_1^2/\tau^2+i\omega t_1} = \Delta(\omega)2\pi$$

and if we replace in the time evolution operator we will get:

$$U_I^{(m)}(t) = \frac{(-i\pi g' a^\dagger)^m}{m!}\Delta^m(\omega)\left[1+erf(t/\tau - i\omega\tau/2)\right]^m$$

From this we can get the matrix element for the transition when $m > 0$:

$$\begin{aligned}\langle n+m| U^{(m)}(t) |n\rangle &= \langle n+m| U_I^{(m)}(t) |n\rangle e^{i(E_n t_0 - E_{n+m}t)/\hbar} \\ &= e^{i(E_n t_0 - E_{n+m}t)/\hbar}\sqrt{\frac{(n+m)!}{n!}}\frac{(-i\pi g')^m}{m!}\Delta^m(\omega)\left[1+erf(t/\tau - i\omega\tau/2)\right]^m\end{aligned}$$

One can follow a similar procedure for $-n \le m < 0$ in which case only the annihilation operators contribute in the dominant order. The result is then:

$$\begin{aligned}\langle n+m| U^{(m)}(t) |n\rangle &= \langle n+m| U_I^{(m)}(t) |n\rangle e^{i(E_n t_0 - E_{n+m}t)/\hbar} \\ &= e^{i(E_n t_0 - E_{n+m}t)/\hbar}\sqrt{\frac{n!}{(n-m)!}}\frac{(-i\pi g')^{|m|}}{|m|!}\Delta^{|m|}(\omega)\left[1+erf(t/\tau - i\omega\tau/2)\right]^{|m|}\end{aligned}$$

5)

This interaction changes adiabatically the strength of the confining potential.

6)

In the interaction picture the perturbation will become:

$$\begin{aligned}H_I(t) &= \gamma e^{-t^2/\tau^2}\frac{\hbar}{2m\omega}\left(a e^{-i\omega t} + a^\dagger e^{i\omega t}\right)^2 \\ &= \hbar\gamma' e^{-t^2/\tau^2}\left(a^2 e^{-2i\omega t} + a^{\dagger 2} e^{2i\omega t} + 1 + 2a^\dagger a\right)\end{aligned}$$

At each step the interaction increases the energy by two, it decreases it by two or it leaves it exactly the same. Therefore m must be an even number and the first non vanishing perturbative order is $m/2$. To get the dominant contribution in the case $m > 0$ we only consider the part of the interaction with the creation operators. For $-n \le m < 0$ we consider only the annihilation operators. The evaluation of the dominant order is similar to the one in part (4) with the substitutions $\omega \to 2\omega$, $m \to m/2$ and $g' \to \gamma'$.

2 Ionization of the Hydrogen atom

(1)

Since there is no other proffered direction in the problem, lets choose the "z" axis of the spherical coordinates to be across $\vec{e}_x$. The time dependent perturbation is:

$$H_1(t) = -zeEe^{\eta t}\cos\omega t$$

where $-e$ is the electron charge. We will consider first order perturbation theory since the magnitude of the higher order corrections is sub-dominant and it does not determine which transition is more probable (unless there is a tie in first order). We will use Fermi's Golden rule for Harmonic perturbations (first lecture notes, page 22). The transition rate between the initial states $|i\rangle = |2,l,m\rangle$ and the final states $|f\rangle = |n',l',m'\rangle$ is given by:

$$\Gamma_{i\to f} = \frac{\pi e^2 E^2}{2\hbar}\left|\langle f|\,\hat{z}\,|i\rangle\right|^2\left[\delta\left(E_{fi}+\hbar\omega\right)+\delta\left(E_{fi}-\hbar\omega\right)\right]$$

Bear in mind that this is the behaviour in the limiting case $\eta \to 0^+$. For finite η we consider the delta functions as Lorentzians with η being the width. This formula suggests that for a given frequency ω the atom will be excited to a particular energy level such that the energy conservation is satisfied with a certain tolerance η. Another point is that the lifetime τ of the initial state is:

$$\tau^{-1} = \sum_f \Gamma_{i\to f}$$

So we need to compare the matrix elements $\langle f|\,\hat{z}\,|i\rangle$ of the various possible transitions. This matrix element can be expressed in spherical coordinates as:

$$\langle f|\,\hat{z}\,|i\rangle = \int_0^\infty dr R_{nl}(r)R_{n'l'}(r)r^3\int d\Omega Y_{lm}(\hat{n})Y_{l'm'}(\hat{n})\sin\theta\cos\theta$$

where the primed indices correspond to the final states.

For the spherical part the following transition rules apply:

$$\begin{aligned}\Delta m &= 0\\ \Delta l &= \pm 1\end{aligned}$$

If we take this into account we have the following possibilities:

1. The $(l=0, m=0)$ states can only absorb a photon and reach any state with $(l=1, m=0)$. The contribution of the spherical part to the matrix element is $1/\sqrt{3}$.
2. The $(l=1, m=0)$ states can absorb a photon and reach any state with $(l=0,2, m=0)$ or they can emit a photon and reach the ground state. The contribution is $1/\sqrt{3}$ for final $l=0$ and $2/\sqrt{15}$ for $l=2$.

3. The $(l=1, m=\pm 1)$ states can only absorb a photon and reach any state with $(l=2, m=\pm 1)$. In both cases the contribution is $1/\sqrt{5}$.

Therefore in total there are only 5 possible combination of the angular momentum.

Because the radial part peaks at radii that increase as n increases and falls exponentially away from the peak, in order for the two radial parts to have a substantial overlap their energies must be close. Therefore for the same l the maximum amplitude occurs for the transitions $2 \rightarrow 3$ or $2 \rightarrow 1$. We remind our selves that:

$$
\begin{aligned}
R_{10} &= \frac{2}{a_0^{3/2}} e^{-r/a_0} \\
R_{20} &= \frac{1}{(2a_0)^{3/2}} \left(2 - \frac{r}{a_0}\right) e^{-r/2a_0} \\
R_{21} &= \frac{1}{(2a_0)^{3/2}} \frac{1}{\sqrt{3}} \frac{r}{a_0} e^{-r/2a_0} \\
R_{30} &= 2\frac{1}{(3a_0)^{3/2}} \left(1 - \frac{2}{3}\frac{r}{a_0} + \frac{2}{27}\frac{r^2}{a_0}\right) e^{-r/3a_0} \\
R_{31} &= \frac{1}{(3a_0)^{3/2}} \frac{4\sqrt{2}}{9} \left(1 - \frac{1}{6}\frac{r}{a_0}\right) \frac{r}{a_0} e^{-r/3a_0} \\
R_{32} &= \frac{1}{(3a_0)^{3/2}} \frac{2\sqrt{2}}{27\sqrt{5}} \frac{r^2}{a_0^2} e^{-r/3a_0}
\end{aligned}
$$

For convenience I will now set $a_0 = 1$. The only possible transitions and the corresponding matrix elements are:

1. $(2,0,0) \rightarrow (3,1,0)$: Spherical part: $1/\sqrt{3}$, radial part: $\frac{27648\sqrt{3}}{15625}$, total: 1.7695.

2. $(2,1,\pm 1) \rightarrow (3,2,\pm 1)$: Spherical part: $1/\sqrt{5}$, radial part: $\frac{165888}{15625\sqrt{5}}$, total: 2.12337.

3. $(2,1,0) \rightarrow (1,0,0)$: Spherical part: $1/\sqrt{3}$, radial part: $\frac{128}{81}\sqrt{\frac{2}{3}}$, total: 0.744936.

4. $(2,1,0) \rightarrow (3,0,0)$: Spherical part: $1/\sqrt{3}$, radial part: $\frac{10368\sqrt{2}}{15625}$, total: 0.541788.

5. $(2,1,0) \rightarrow (3,2,0)$: Spherical part: $2/\sqrt{15}$, radial part: $\frac{165888}{15625\sqrt{5}}$, total: 2.45185.

We see that the transitions $2 \rightarrow 3$ are more likely than $2 \rightarrow 1$. Also we note that the transition rates are: $\tau_{200}^{-1} : \tau_{210}^{-1} : \tau_{21\pm 1}^{-1} = 1.77 : 2.99 : 2.12$=26%:43%:31%, where the lifetime of $(2,1,0)$ was obtained by summing over the two possible transitions (the other states have only one possible transition). The state $(2,1,0)$ is the most short lived and the $(2,0,0)$ the longest lived. That conclusion does not depend on the choice of the z axis.

(2)

The transition is between the ground state of the Hydrogen atom:

$$\langle\vec{r}|n\rangle = \langle\vec{r}|\psi_0\rangle = \frac{e^{-r/a_0}}{a_0^{3/2}\sqrt{\pi}}$$

with energy E_0 and any plane wave:

$$\langle\vec{r}|m\rangle = \left\langle\vec{r}|\vec{k}\right\rangle = \frac{1}{\sqrt{V}}e^{i\vec{k}\cdot\vec{r}}$$

with energy $E(\vec{k}) = \frac{\hbar^2\vec{k}^2}{2m}$. V is the volume of the universe. The transition matrix element between the two is:

$$\begin{aligned} M(\vec{k}) = \langle m| - eE\hat{z}\,|n\rangle &= -\frac{eE}{\sqrt{V}}\frac{1}{a_0^{3/2}\sqrt{\pi}}\int d^3\vec{r} z e^{-r/a_0}e^{i\vec{k}\cdot\vec{r}} \\ &= -\frac{eE}{\sqrt{V}}\frac{1}{a_0^{3/2}\sqrt{\pi}}\frac{\partial}{i\partial k_z}\int d^3\vec{r} e^{-r/a_0}e^{i\vec{k}\cdot\vec{r}} \end{aligned}$$

In the last integral we can use spherical coordinates with $\vec{k}$ as the z direction:

$$\begin{aligned} \int d^3\vec{r} e^{-r/a_0}e^{i\vec{k}\cdot\vec{r}} &= 2\pi\int_0^\infty dr\int_0^\pi d\theta r^2\sin\theta e^{-r/a_0}e^{ikr\cos\theta} \\ &= 4\pi\int_0^\infty dr e^{-r/a_0}r^2\frac{\sin kr}{kr} \\ &= \frac{8\pi a_0^3}{\left(1+a_0^2k^2\right)^2} \end{aligned}$$

The modulus of of the transition amplitude squared is:

$$\left|M(\vec{k})\right|^2 = \frac{e^2E^2}{V}a_0^5\pi\left[\frac{32a_0k_z}{\left(1+a_0^2k^2\right)^3}\right]^2$$

From Fermi's golden rule for Harmonic perturbations the total ionization rate is the transition rate to all scattering states combined:

$$\tau^{-1}(\omega) = \frac{1}{4}\frac{2\pi}{\hbar}\int\frac{Vd^3\vec{k}}{(2\pi)^3}\left[\delta\left(E(\vec{k})+\hbar\omega-E_0\right)+\delta\left(E(\vec{k})-\hbar\omega-E_0\right)\right]\left|M(\vec{k})\right|^2$$

Since (without loss of generality) $\omega > 0$ and also $E(\vec{k}) > 0$ and $-E_0 > 0$ the first delta function vanishes and replacing the matrix element we get:

$$\tau^{-1}(\omega) = \frac{1}{4}\frac{1024e^2E^2a_0^5}{4\pi\hbar}\int d^3\vec{k}\delta\left(E(\vec{k}) - \hbar\omega - E_0\right)\frac{a_0^2k_z^2}{(1+a_0^2k^2)^6}$$

To proceed we use spherical coordinates:

$$\begin{aligned} d^3\vec{k} &= k^2dkd\Omega = \frac{1}{2}kd(k^2)d\Omega = \frac{1}{2}\left(\frac{2m}{\hbar^2}\right)^{3/2}E^{1/2}dEd\Omega \\ k_z &= k\cos\theta \end{aligned}$$

In the energy integral the delta function will simply set $E = \hbar\omega + E_0$ and $a_0^2k^2 = 2ma_0^2E/\hbar^2 = E/E_b$:

$$\tau^{-1}(\omega) = \frac{1}{4}\frac{1024e^2E^2ma_0^4}{4\pi\hbar^3}\frac{(\hbar\omega+E_0)^{3/2}/E_b^{3/2}}{(1+(\hbar\omega+E_0)/E_b)^6}\int d\Omega\cos^2\theta$$

From this expression we can tell that the angular distribution of the electrons is $\cos^2\theta$.

Performing the angular integration (remember that $d\Omega = \sin\theta d\theta d\phi$) we get a factor of $4\pi/3$ and the total transition rate will become:

$$\tau^{-1}(\omega) = \frac{1}{4}\frac{1024e^2E^2ma_0^4}{3\hbar^3}\frac{(\hbar\omega+E_0)^{3/2}/E_b^{3/2}}{(1+(\hbar\omega+E_0)/E_b)^6} = \frac{256e^2E^2ma_0^4}{3\hbar^3}f\left(\frac{\hbar\omega+E_0}{E_b}\right)$$

where the function $f(x)$ is plotted in Fig. 1. Most emitted electrons will have energy close to $0.51E_b = 0.51\frac{\hbar^2}{2ma_0^2}$ and they will have a wavevector $k \approx 0.72a_0^{-1}$ which is approximately the Bohr radius.

Finally we have to note here that in order for the emitted electrons to really be plane waves despite the attractive potential of the hydrogen atom their wavelength must be much larger than a_0^{-1}. This means that $E_b \ll E(\vec{k})$.

3 Instantaneous Perturbation

Using the Born approximation for initial time 0^- and final 0^+ is essentially like setting $t = 0$ which will give:

$$\langle f|i\rangle = -\frac{i}{\hbar}\langle f|\hat{V}|i\rangle$$

4 Coulomb Excitation

1)

In this problem, a heavy charged particle comes from very far, it passes near a hydrogen atom, it makes a relatively small mess (that is why we use perturbation

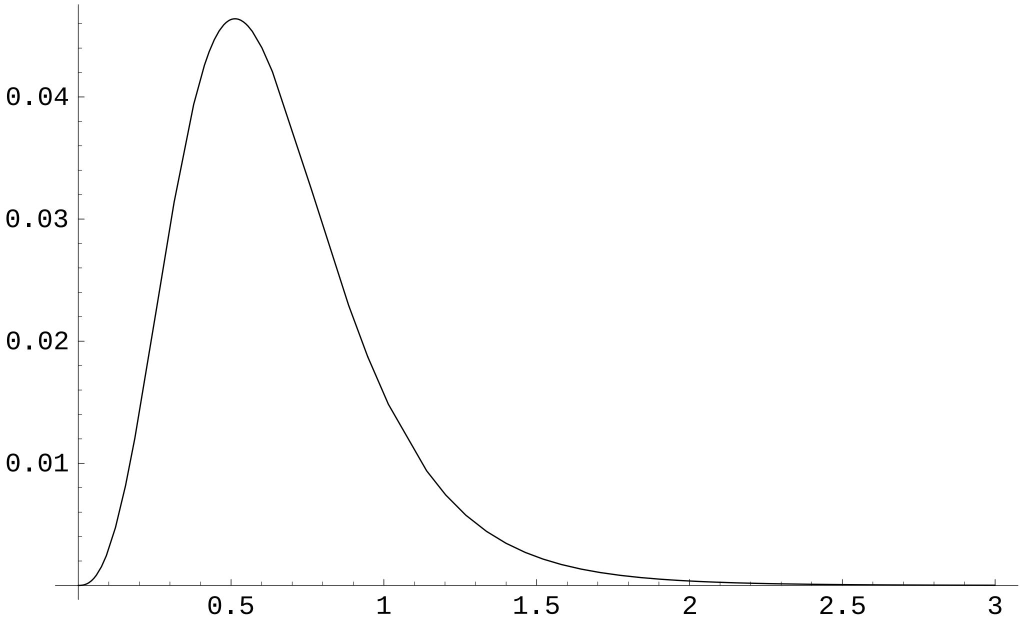

Figure 1: The function $f(x) = \frac{x^2}{(1+x^2)^6}$. The difference from the transition rate is that the center is at zero instead of ω_0/ω_b.

theory) and it runs away to infinity leaving the hydrogen atom in a non clean state.

The perturbation is just the Coulomb interaction between the heavy particle and the electron of the atom:

$$H_1(t) = -\frac{Ze^2}{\left|\hat{\vec{r}} - \vec{R}(t)\right|}$$

Note that $\vec{r}$ as it appears in this equation is an operator.

2)

We will use the Born approximation (lecture notes, 12th page):

$$P_{n\to m} = \frac{1}{\hbar^2}\left|\int_{-\infty}^{\infty} dt e^{\frac{i}{\hbar}(E_m - E_n)t} \langle m| H_1(t) |n\rangle\right|^2$$

3)

If the characteristic mean square radius of the orbital is much smaller than d then $|\vec{r}| \ll \left|\vec{R}(t)\right|$ and we can keep only linear terms in $\vec{r}$:

$$\left|\vec{r} - \vec{R}(t)\right|^{-1/2} \approx \left(\vec{R}^2(t) - 2\vec{r}\cdot\vec{R}(t)\right)^{-1/2} \approx \frac{1}{\left|\vec{R}(t)\right|} + \frac{\vec{r}\cdot\vec{R}(t)}{\left|\vec{R}(t)\right|^3}$$

The first term only shifts the total energy but does not induce transitions (it is a constant so its matrix element is zero) so it will be dropped. Replacing this in the Born approximation will give:

$$P_{n\to m} = \frac{Z^2e^4}{\hbar^2}\left|\int_{-\infty}^{\infty} dt \frac{utx_{mn} + dy_{mn}}{(u^2t^2 + d^2)^{\frac{3}{2}}} e^{i\omega_{mn}t}\right|^2$$

where $x_{mn} = \langle m| \hat{x} |n\rangle$, $y_{mn} = \langle m| \hat{y} |n\rangle$ and $\omega_{mn} = \frac{E_m - E_n}{\hbar}$.

4)

The characteristic times scale of the perturbation is:

$$\tau = \frac{d}{u}$$

We can write:

$$\left(u^2t^2 + d^2\right)^{\frac{3}{2}} = u^3\left(t^2 + \tau^2\right)^{\frac{3}{2}}$$

5)

Notice that the given integrals are from 0 to $+\infty$ and so we have to break the original integral:

$$\int_{-\infty}^{\infty} dt f(t) = \int_{0}^{\infty} dt \left[f(t) + f(-t)\right]$$

So that the probability becomes:

$$P_{n\to m} = \frac{4Z^2e^4}{\hbar^2 u^4}\left|\int_0^\infty dt \frac{itx_{mn}\sin\left(\omega_{mn}t\right) + \tau y_{mn}\cos\left(\omega_{mn}t\right)}{\left(t^2+\tau^2\right)^{\frac{3}{2}}}\right|^2$$

Using the given integrals:

$$P_{n\to m} = \frac{4Z^2e^4}{\hbar^2 u^4}\left|ix_{mn}\omega_{mn}K_0\left(\omega_{mn}\tau\right) - y_{mn}\frac{\partial}{\partial\tau}K_0\left(\omega_{mn}\tau\right)\right|^2$$

If the particle moves too slow, $\omega_{mn}\tau \ll 1$, we can approximate the Bessel function as:

$$\begin{aligned} K_0\left(\omega_{mn}\tau\right) &= \ln\frac{2}{\omega_{mn}\tau} \\ -\frac{\partial}{\partial\tau}K_0\left(\omega_{mn}\tau\right) &= \frac{1}{\tau} \end{aligned}$$

From this we get the probability:

$$P_{n\to m} = \frac{4Z^2e^4}{\hbar^2 d^2 u^2}\left(x_{mn}^2\tau^2\omega_{mn}^2\ln^2\frac{2}{\omega_{,mn}\tau} + y_{mn}^2\right)$$