Physics 580: Quantum Mechanics I
Department of Physics, UIUC
Fall Semester 2006
Professor Eduardo Fradkin

Problem Set No. 5:
Quantum Mechanics in two and three dimensions
Due Date: October 30

1 Charged Particle in an Uniform Magnetic Field

In this problem you will consider a particle of mass M and charge e, restricted to move on a large disk of radius R in the presence of a magnetic field of strength B, perpendicular to the plane. The magnetic flux through the rectangle is $\Phi = \pi R^2 B$. You will work in the circular gauge in which the components of the vector potential, for $B > 0$, are

$$A_x = -\frac{1}{2} B y, \qquad A_y = \frac{1}{2} B x$$

In this gauge it is convenient to work in polar coordinates

$$x = r\cos\varphi, \qquad y = r\sin\varphi$$

where

$$0 \leq r < \infty, \qquad \text{and} \qquad 0 \leq \varphi < 2\pi$$

In polar coordinates the components of the vector potential become

$$A_r = 0, \qquad \text{and} \qquad A_\varphi = \frac{B}{2} r$$

You will work in the limit of a large disk, *i. e.* $R \to \infty$, while keeping the total flux Φ fixed and finite.

The Hamiltonian for this problem is

$$\hat{H} = \frac{1}{2M}\left(\hat{\Pi}_x^2 + \hat{\Pi}_y^2\right)$$

where

$$\hat{\Pi}_x = \hat{P}_x - \frac{e}{c} A_x(\hat{X}, \hat{Y}) \ , \qquad \hat{\Pi}_y = \hat{P}_y - \frac{e}{c} A_y(\hat{X}, \hat{Y})$$

$$A_x(\hat{X}, \hat{Y}) = -\frac{B}{2}\hat{Y} \ , \qquad A_y(\hat{X}, \hat{Y}) = \frac{B}{2}\hat{X}$$

The cartesian components of the canonical momentum, $\hat{P}_x$ and $\hat{P}_y$, and cartesian components of the coordinate, $\hat{X}$ and $\hat{Y}$, obey the canonical commutation relations:

$$\left[\hat{X}, \hat{P}_x\right] = \left[\hat{Y}, \hat{P}_y\right] = i\hbar \ , \qquad \left[\hat{X}, \hat{P}_y\right] = \left[\hat{Y}, \hat{P}_x\right] = 0$$

1. Show that the z-component of the angular momentum $\hat{L}_z = \hat{X}\hat{P}_y - \hat{Y}\hat{P}_x$ is conserved. Find an expression for $\hat{L}_z$ in polar coordinates. Derive the Schrödinger Equation for the eigenstates of energy E in the polar coordinates (r, φ).

2. Show that in polar coordinates the eigenstates of $\hat{H}$ have the form

$$\Psi(r, \varphi) = \frac{1}{\sqrt{2\pi}} \, e^{im\varphi} \, R_m(r)$$

   Explain the physical meaning of the quantum number m and find the allowed values of m. Find the Schrödinger Equation for the radial wave function $R_m(r)$. Write this equation in terms of the cyclotron frequency $\omega_c = eB/Mc$ and of the magnetic length $\ell_0 = \sqrt{\hbar c/eB}$. Determine the boundary conditions for $R_m(r)$ as $r \to 0$ and $r \to \infty$.

3. The radial wave functions $R_m(r)$ of the allowed eigenstates have the form

$$R_m(r) = e^{-\dfrac{u}{2}} \, u^{|m|/2} \, F(u)$$

   where $u = r^2/(2\ell_0^2)$ and $F(u) = F(\alpha, \gamma, u)$ is a confluent hypergeometric function

$$F(\alpha, \gamma, u) = 1 + \frac{\alpha}{\gamma}\frac{u}{1!} + \frac{\alpha(\alpha+1)}{\gamma(\gamma+1)}\frac{u^2}{2!} + \ldots$$

   which satisfies the differential equation

$$u \, \frac{d^2F}{du^2} + (\gamma - u) \, \frac{dF}{du} - \alpha F = 0$$

   Show that for this problem the parameters α and γ are

$$\alpha = \frac{1}{2}\,(-m + |m| + 1) - \frac{E}{\hbar\omega_c} \qquad \text{and} \qquad \gamma = |m| + 1$$

   Show that these wave functions satisfy the boundary conditions at $r = 0$ and as $r \to \infty$ if the function $F(u)$ is a polynomial of degree n, with $n = 0, 1, 2, \ldots$; hence, $\alpha = -n$. Use these results to find the allowed energy levels (the Landau levels).
   Note: these polynomials are the generalized Laguerre polynomials

$$F(-n, |m|+1, u) = L_n^{|m|+1}(u) = (-1)^{|m|} \frac{|m|!}{(n+|m|)!} e^u \frac{d^{n+|m|}}{du^{n+|m|}} (e^{-u} u^n)$$

   How does the degeneracy of the Landau levels show up in this solution?

4. Consider now the wave functions of the states with $n = 0$ and $m \leq 0$. Calculate their energies. Plot their probability distributions as a function if r and φ for $m = 0$ and $m = -1$. Calculate the expectation value $\langle 0, m|\hat{X}^2 + \hat{Y}^2|0, m\rangle$. What is the r. m. s. value of the position vector in these states? How does it depend on ℓ_0? How does in general depend on m?

5. Show that for a particle in a state at time t with wave function $\Psi(x, y, t)$ the current density $\vec{J}(x, y, t)$ and probability density $|\Psi(x, y, t)|^2$ satisfy the continuity equation

$$\frac{\partial |\Psi|^2}{\partial t} + \vec{\bigtriangledown} \cdot \vec{J} = 0$$

where the current is

$$\vec{J} = \frac{\hbar}{2Mi} \left(\Psi^* \vec{D}\Psi - \left(\vec{D}\Psi\right)^* \Psi \right)$$

where

$$\vec{D} = \vec{\bigtriangledown} + i\frac{e}{\hbar c}\vec{A}(x, y)$$

is called the covariant derivative. Find a general expression for the radial and azimuthal components $J_r(r, \varphi)$ and $J_\varphi(r, \varphi)$ of the current, and use this result to calculate J_r and J_φ for the eigenstates $|n, m\rangle = |0, 0\rangle, |0, -1\rangle$. Give a physical interpretation of your results.

2 Angular Momentum in Three Dimensions

In this problem you will work out a number of important properties of the angular momentum operators and of their eigentstates.

1. Construct the three 4×4 matrices that represent the operators $\hat{J}_x$, $\hat{J}_y$ and $\hat{J}_z$. Show that these matrices obey the commutation relations of the angular momentum operators. Construct the 4×4 matrices which represent the operators $\hat{J}_+$, $\hat{J}_-$ and $\hat{J}^2$.

2. Consider now the general angular momentum eigenstates $|j, m\rangle$. Show that

$$\langle j, m|\hat{J}_x|j, m\rangle = \langle j, m|\hat{J}_y|j, m\rangle = 0$$

and that

$$\langle j, m|\hat{J}_x^2|j, m\rangle = \langle j, m|\hat{J}_y^2|j, m\rangle = \hbar^2[j(j + 1) - m^2]/2$$

3. Use these results to show that $\Delta J_x \, \Delta J_y$ satisfies an Uncertainty Principle. Show that the bound is saturated by the states $|j, \pm j\rangle$.

4. Let us define a rotation $R(\alpha,\beta,\gamma)$ where α, β and γ are the *Euler Angles* for three successive rotations:

$$U[R(\alpha,\beta,\gamma)] = e^{-i\alpha\hat{J}_z/\hbar}\, e^{-i\beta\hat{J}_y/\hbar}\, e^{-i\gamma\hat{J}_z/\hbar}$$

Construct the matrix $D^{(1)}[R(\alpha,\beta,\gamma)]$ explicitly as a product of three matrices. Let $\Psi = D^{(1)}|1,1\rangle$. Show that

$$\langle\Psi|\vec{J}|\Psi\rangle = \hbar(\sin\beta\cos\alpha\;\vec{e}_x + \sin\beta\sin\alpha\;\vec{e}_y + \cos\beta\;\vec{e}_z)$$

where $\vec{e}_x$, $\vec{e}_y$ and $\vec{e}_z$ are unit vectors along the directions x, y and z. Show that for no value of α, β and γ it is possible to rotate $|1,1\rangle$ into just $|1,0\rangle$. Show that it is always possible to rotate $|1,m\rangle$ into a linear combination involving $|1,m'\rangle$, that is

$$\langle 1,m|D^{(1)}[R(\alpha,\beta,\gamma)]|1,m'\rangle \neq 0$$

for some α, β, γ and any m and m'.

3 The Angular Momentum states in the Coordinate Basis

In what follows we will denote the wave functions of the angular momentum states in spherical coordinates as

$$\langle\theta,\phi|\ell,m\rangle = Y_\ell^m(\theta,\phi)$$

where $-\pi \leq \theta \leq \pi$ and $0 \leq \varphi < 2\pi$. As usual, $\ell = 0,1,2,\ldots$ and $|m| \leq \ell$.

1. Show that in spherical coordinates, the operators $\hat{\vec{L}}^2$, $\hat{L}_z$ and $\hat{L}_\pm$ are given by

$$\hat{\vec{L}}^2 = -\hbar^2\left(\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\sin\theta\frac{\partial}{\partial\theta} + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2}\right)$$

$$\hat{L}_z = -i\hbar\frac{\partial}{\partial\phi}$$

$$\hat{L}_\pm = \pm\hbar e^{\pm i\phi}\left(\frac{\partial}{\partial\theta} \pm i\cot\theta\frac{\partial}{\partial\phi}\right)$$

2. Show that the wave function $Y_\ell^\ell(\theta,\phi)$ is the solution of the equation

$$\left(\frac{\partial}{\partial\theta} + i\cot\theta\frac{\partial}{\partial\phi}\right)Y_\ell^\ell(\theta,\phi) = 0$$

What condition on the ket $|\ell,\ell\rangle$ does this equation represent?

3. Using the lowering operator $\hat{L}_-$, construct the wave function for the ket $|\ell,\ell-2\rangle$, including the normalization constant.