1 I Charged Particle in a Uniform Magnetic Field

(1)

$$\begin{aligned}
[\hat{L}_z, H] &= [\hat{X}\hat{P}_y - \hat{Y}\hat{P}_x, \frac{1}{2M}(\hat{P}_x + \frac{eB}{2c}\hat{Y})^2 + \frac{1}{2M}(\hat{P}_y - \frac{eB}{2c}\hat{X})^2] \\
&= [\hat{X}\hat{P}_y - \hat{Y}\hat{P}_x, \frac{1}{2M}(\hat{P}_x + \hat{P}_y)^2 + \frac{eB}{4Mc}(\hat{P}_x\hat{Y} + \hat{Y}\hat{P}_x - \hat{P}_y\hat{X} - \hat{X}\hat{P}_y) + \frac{e^2B^2}{8Mc^2}(\hat{X}^2 + \hat{Y}^2)] \\
&= \frac{1}{2M}([\hat{X}, \hat{P}_x^2]\hat{P}_y - [\hat{Y}, \hat{P}_y^2]\hat{P}_x) + \frac{eB}{4Mc}(2[\hat{X}\hat{P}_y, \hat{P}_x\hat{Y} + 2[\hat{Y}\hat{P}_x, \hat{P}_y\hat{X}]) \\
&\quad + \frac{e^2B^2}{8Mc^2}([\hat{X}\hat{P}_y, \hat{Y}^2] - [\hat{Y}\hat{P}_x, \hat{X}^2]) \\
&= \frac{1}{2M}(2i\hbar\hat{P}_x\hat{P}_y - 2i\hbar\hat{P}_y\hat{P}_x) + \frac{eB}{4Mc}[2i\hbar(\hat{Y}\hat{P}_y - \hat{X}\hat{P}_x) - 2i\hbar(\hat{Y}\hat{P}_y - \hat{X}\hat{P}_x)] + \frac{e^2B^2}{8Mc^2}(-2i\hbar\hat{X}\hat{Y} - (-2i\hbar\hat{Y}\hat{X})) \\
&= 0
\end{aligned}$$

Thus L_z is conserved. As for its expression in polar coordinates:

$$\left\{\begin{array}{c} r = \sqrt{x^2 + y^2} \\ \theta = arctg\frac{y}{x} \end{array}\right. \Rightarrow \left\{\begin{array}{c} \frac{\partial r}{\partial x} = \frac{x}{r};\ \frac{\partial r}{\partial y} = \frac{y}{r} \\ \frac{\partial \theta}{\partial x} = \frac{-y}{r^2};\ \frac{\partial \theta}{\partial y} = \frac{x}{r^2} \end{array}\right.$$

$$\begin{aligned}
L_z &= \hat{X}\hat{P}_y - \hat{Y}\hat{P}_x = -i\hbar(x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}) \\
&= -i\hbar[x(\frac{\partial}{\partial r}\frac{\partial r}{\partial y} + \frac{\partial}{\partial \theta}\frac{\partial \theta}{\partial y}) - y(\frac{\partial}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial}{\partial \theta}\frac{\partial \theta}{\partial x})] \\
&= i\hbar(yx - xy)\frac{1}{r}\frac{\partial}{\partial r} + i\hbar(-y^2 - x^2)\frac{1}{r^2}\frac{\partial}{\partial \theta} \\
&= -i\hbar\frac{\partial}{\partial \theta}
\end{aligned}$$

The schrodinger equation for the eigenstates of energy E is $\hat{H}\Psi = E\Psi$, where

$$\begin{aligned}
\hat{H} &= \frac{1}{2M}(\hat{\Pi}_x^2 + \hat{\Pi}_y^2) \\
&= \frac{1}{2M}(\hat{P}_x^2 + \hat{P}_y^2) + \frac{eB}{2Mc}(\hat{Y}\hat{P}_x - \hat{X}\hat{P}_y) + \frac{e^2B^2}{8Mc^2}(\hat{X}^2 + \hat{Y}^2) \\
&= \frac{-\hbar^2}{2M}(\frac{d^2}{dx^2} + \frac{d^2}{dy^2}) - \frac{eB}{2Mc}\hat{L}_z + \frac{e^2B^2}{8Mc^2}r^2 \\
&= \frac{-\hbar^2}{2M}(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}) + \frac{i\hbar eB}{2Mc}\frac{\partial}{\partial \theta} + \frac{e^2B^2}{8Mc^2}r^2
\end{aligned}$$

Thus the Schrodinger equation is $\frac{-\hbar^2}{2M}(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2})\Psi + \frac{i\hbar eB}{2Mc}\frac{\partial \Psi}{\partial \theta} + \frac{e^2B^2}{8Mc^2}r^2\Psi = E\Psi$

(2) Since $\hat{L}_z$ is conserved, l_z should be a good quantum number for the system. Meanwhile,its eigen-equation

is $\hat{L}_z\phi = -i\hbar\frac{\partial}{\partial\psi}\phi = m\hbar\phi$, which satisfies $\phi(0) = \phi(2\pi)$. Then $\phi_m = e^{im\psi}$, where m is only integer, and we can write the eigenstates of $\hat{H}$ in the form $\Psi(r,\psi) = \frac{1}{\sqrt{2\pi}}e^{im\psi}R_m(r)$. Take this form into the Shrodinger equation and we can get the equation for the radial wave function:

$$\frac{d^2}{dr^2}R_m(r) + \frac{1}{r}\frac{d}{dr}R_m(r) + (\frac{2E}{\hbar\omega_c l_0^2} + \frac{m}{l_0^2} - \frac{r^2}{4l_0^4} - \frac{m^2}{r^2})R_m(r) = 0 \tag{1}$$

where $\omega_c = eB/Mc, l_0 = \sqrt{\hbar c/eB}$.
As for the boundary condition, when $r \longrightarrow 0$, due to the single value of $\Psi(0,\psi)$ for different ψ,$R_m(0) = 0$; when $r \longrightarrow \infty$, $\Psi(r,\psi) = 0, thus\ R(\infty) = 0$

(3)

$$\frac{d}{dr} = \frac{r}{l_0^2}\frac{d}{d\mu};\ \frac{d^2}{dr^2} = \frac{1}{l_0^2}\frac{d}{d\mu} + \frac{2\mu}{l_0^2}\frac{d^2}{d\mu^2} \tag{2}$$

Take the equality (2) into the radial equation (1), we can get:

$$\mu\frac{d^2}{d\mu^2}R_m(r) + \frac{d}{d\mu}R_m(r) + (\frac{E}{\hbar\omega_c} + \frac{m}{2} - \mu - \frac{m^2}{4\mu})R_m(r) = 0 \tag{3}$$

If we set $R_m(r) = e^{\frac{\mu}{2}}\mu^{|m|/2}F(\mu)$, the equation (3) will become:

$$\mu\frac{d^2F}{d\mu^2} + (|m| + 1 - \mu)\frac{dF}{d\mu} - [\frac{1}{2}(|m| - m + 1) - \frac{E}{\hbar\omega_C}]F(\mu) = 0 \tag{4}$$

which satisfies the differential equation

$$\mu\frac{d^2F}{d\mu^2} + (\gamma - \mu)\frac{dF}{d\mu} - \alpha F = 0 \tag{5}$$

and the parameters $\gamma = |m| + 1, and\ \alpha = \frac{1}{2}(-m + |m| + 1) - \frac{E}{\hbar\omega_c}$. The solution for F(u) is the confluent hypergeometric function $F(\alpha,\gamma,\mu) = 1 + \frac{\alpha}{\gamma}\frac{\mu}{1!} + \frac{\alpha(\alpha+1)}{\gamma(\gamma+1)}\frac{\mu^2}{2!} + \ldots$ Only when $F(\mu)$ is a polynomial of degree n, the boundary conditions are satisfied:

$$R_m \propto \begin{cases} \mu^{|m|/2} \longrightarrow 0 & \text{when } r \longrightarrow 0 \\ \frac{\mu^{|m|/2+n}}{e^{\mu/2}} = 0 & \text{when } r \longrightarrow \infty \end{cases}$$

In order to make $F(\mu)$ a polynomial of degree n, the term $\frac{\alpha(\alpha+1)\ldots(\alpha+n)}{\gamma(\gamma+1)\ldots(\gamma+n)}\frac{u^{n+1}}{(n+1)!}$ should be zero and then we can get $\alpha + n = 0$, which means:

$$\frac{1}{2}(-m + |m| + 1) - \frac{E}{\hbar\omega_c} + n = 0$$

Thus the allowed energy should be quantized and

$$E = \frac{\hbar\omega_c}{2}(|m| - m + 1 + 2n) = \begin{cases} \hbar\omega_c(\frac{1}{2} + n) & \text{when } m \geq 0 \\ \hbar\omega_c(-m + n + \frac{1}{2}) & \text{when } m < 0 \end{cases}$$

The Landau levels show up to be degenerate for $m \geq 0$ with the same n.

(4) For the states with $n = 0; m \leq 0$, the energy is $E = \hbar\omega_c(-m + \frac{1}{2})$.
Then $\alpha = 0, \gamma = -m + 1, \therefore F(\alpha, \gamma, \mu) = 1; \Psi(r, \theta) = \frac{1}{\sqrt{2\pi}} e^{im\phi} e^{-u} u^{|m|/2}$.
After normalization, $\Psi = \frac{1}{\sqrt{2\pi\Gamma(-m+1)l_0}} e^{im\phi} e^{-u/2} u^{|m|/2}$

$$P(r, \phi) = \langle\Psi|\Psi\rangle = \begin{cases} \frac{1}{2\pi l_0^2} e^{-r^2/2l_0^2} & \text{when } m = 0 \\ \frac{1}{2\pi l_0^2} e^{-r^2/2l_0^2}(r^2/2l_0^2) & \text{when } m = -1 \end{cases} \tag{6}$$

It is easy to show that $\langle x\rangle = \langle y\rangle = 0$ and $\langle x^2\rangle = \langle y^2\rangle = \frac{1}{2}\langle r^2\rangle$, meanwhile,

$$\begin{aligned} \langle 0, m|\hat{X}^2 + \hat{Y}^2|0, m\rangle &= \frac{1}{2\pi\Gamma(-m+1)l_0^2} \int_0^{2\pi} d\phi \int_0^{\infty} rdrr^2 e^{-\mu}\mu^{-m} \\ &= \frac{2l_0^4}{\Gamma(-m+1)l_0^2} \int_0^{\infty} d\mu\mu^{-m+1} e^{-mu} \\ &= 2l_0^2\Gamma(-m+2)/\Gamma(-m+1) = 2(-m+1)l_0^2 \end{aligned}$$

(5)For the Schrodinger equation of particles in the magnetic field,

$$\frac{1}{2M}[-\hbar^2 \bigtriangledown^2 + \frac{ie\hbar}{c}(\overrightarrow{A} \cdot \overrightarrow{\bigtriangledown} + \overrightarrow{\bigtriangledown} \cdot \overrightarrow{A}) + \frac{e^2A^2}{c^2}]\Psi = i\hbar\frac{\partial\Psi}{\partial t}$$

We can get two equations from Schrodinger equation:

$$\frac{\Psi^*}{2M}[-\hbar^2 \bigtriangledown^2 + \frac{ie\hbar}{c}(\overrightarrow{A} \cdot \overrightarrow{\bigtriangledown} + \overrightarrow{\bigtriangledown} \cdot \overrightarrow{A}) + \frac{e^2A^2}{c^2}]\Psi = \Psi^* i\hbar\frac{\partial\Psi}{\partial t} \tag{7}$$

$$\frac{\Psi}{2M}[-\hbar^2 \bigtriangledown^2 - \frac{ie\hbar}{c}(\overrightarrow{A} \cdot \overrightarrow{\bigtriangledown} + \overrightarrow{\bigtriangledown} \cdot \overrightarrow{A}) + \frac{e^2A^2}{c^2}]\Psi^* = \Psi(-i\hbar)\frac{\partial\Psi^*}{\partial t} \tag{8}$$

Equation (6)-Equation (7) and then:

$$\frac{-\hbar^2}{2M}(\Psi^* \bigtriangledown^2 \Psi - \Psi \bigtriangledown^2 \Psi^*) + \frac{ie\hbar}{2Mc}[(\Psi^*\overrightarrow{A} \cdot \overrightarrow{\bigtriangledown}\Psi + \Psi\overrightarrow{\bigtriangledown} \cdot \overrightarrow{A}\Psi^*) + (\Psi\overrightarrow{A} \cdot \overrightarrow{\bigtriangledown}\Psi^* + \Psi^*\overrightarrow{\bigtriangledown} \cdot \overrightarrow{A}\Psi)] = i\hbar[\Psi\frac{\partial\Psi^*}{\partial t} + \Psi^*\frac{\partial\Psi}{\partial t}]$$

$$\therefore \frac{-\hbar^2}{2M} \bigtriangledown \cdot(\Psi^* \bigtriangledown \Psi - \Psi \bigtriangledown \Psi^*) + \frac{ie\hbar}{2Mc}\overrightarrow{\bigtriangledown} \cdot (2\Psi^*\Psi\overrightarrow{A}) = i\hbar\frac{\partial}{\partial t}[\Psi\Psi^*]$$

$$\therefore \frac{\partial}{\partial t}[\Psi\Psi^*] + \frac{\hbar}{2Mi}\overrightarrow{\bigtriangledown} \cdot (\Psi^* \bigtriangledown \Psi - \Psi \bigtriangledown \Psi^* - \frac{2ie}{\hbar c}\Psi^*\Psi\overrightarrow{A}) = 0 \tag{9}$$

which is the just the continuity equation $\frac{\partial|\Psi|^2}{\partial t} + \overrightarrow{\bigtriangledown} \cdot \overrightarrow{J} = 0$, where $\overrightarrow{J} = \frac{\hbar}{2Mi}(\Psi^*\overrightarrow{D}\Psi - (\overrightarrow{D}\Psi)^*\Psi)$ and $\overrightarrow{D} = \overrightarrow{\bigtriangledown} - i\frac{e}{\hbar c}\overrightarrow{A}$
Since $\overrightarrow{D} = \overrightarrow{\bigtriangledown} - i\frac{e}{\hbar c}\overrightarrow{A} = (\frac{\partial}{\partial r} - i\frac{e}{\hbar c}A_r(r, \phi))\overrightarrow{e_r} + (\frac{1}{r}\frac{\partial}{\partial\phi} - i\frac{e}{\hbar c}A_\phi(r, \phi))\overrightarrow{e_\phi}$. Thus:

$$\begin{cases} J_r(r, \phi) = \frac{\hbar}{2Mi}(\Psi^*\frac{\partial}{\partial r}\Psi - \Psi\frac{\partial}{\partial r}\Psi^* - \frac{2ie}{\hbar c}A_r(r, \phi)\Psi^*\Psi) \\ J_\phi(r, \phi) = \frac{\hbar}{2Mi}[\frac{1}{r}(\Psi^*\frac{\partial}{\partial\phi}\Psi - \Psi\frac{\partial}{\partial\phi}\Psi^*) - \frac{2ie}{\hbar c}A_\phi(r, \phi))\Psi^*\Psi] \end{cases} \tag{10}$$

In this case,$A_r = 0$ and $A_\phi = \frac{B}{2}r$; for the case $|n, m\rangle = |0, 0\rangle$,$\Psi = \frac{1}{\sqrt{2\pi} l_0} e^{-r^2/4l_0^2}$:

$$\begin{cases} J_r(r, \phi) = 0 \\ J_\phi(r, \phi) = -\frac{\omega_c r}{4\pi l_0^2} e^{-r^2/2l_0^2} \end{cases} \tag{11}$$

For the case $|n, m\rangle = |0, -1\rangle$, $\Psi = \frac{1}{\sqrt{2\pi} l_0} e^{-i\phi} e^{-r^2/4l_0^2} (\frac{r^2}{2l_0^2})^{\frac{1}{2}}$ and then:

$$\begin{cases} J_r(r, \phi) = 0 \\ J_\phi(r, \phi) = -\frac{\hbar}{4\pi M l_0^4} (\frac{r}{2l_0^2} + \frac{1}{r}) e^{-r^2/2l_0^2} r^2 \end{cases} \tag{12}$$

2 II Angular Momentum in Three Dimensions

1.In the z-basis, Obviously,

$$\hat{J}_z = \hbar \begin{pmatrix} 3/2 & 0 & 0 & 0 \\ 0 & 1/2 & 0 & 0 \\ 0 & 0 & -1/2 & 0 \\ 0 & 0 & 0 & -3/2 \end{pmatrix}$$

Using the relation $J_\pm |J, m\rangle = \hbar\sqrt{J(J+1) - m(m \pm 1)} |J, m \pm 1\rangle$, we can get:

$$J_+ = \hbar \begin{pmatrix} 0 & \sqrt{3} & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & \sqrt{3} \\ 0 & 0 & 0 & 0 \end{pmatrix}; J_- = \hbar \begin{pmatrix} 0 & 0 & 0 & 0 \\ \sqrt{3} & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & \sqrt{3} & 0 \end{pmatrix}$$

$$J_x = \frac{J_+ + J_-}{2} = \frac{\hbar}{2} \begin{pmatrix} 0 & \sqrt{3} & 0 & 0 \\ \sqrt{3} & 0 & 2 & 0 \\ 0 & 2 & 0 & \sqrt{3} \\ 0 & 0 & \sqrt{3} & 0 \end{pmatrix}; J_y = \frac{J_+ - J_-}{2i} = \frac{\hbar}{2} \begin{pmatrix} 0 & -\sqrt{3}i & 0 & 0 \\ \sqrt{3}i & 0 & -2i & 0 \\ 0 & 2i & 0 & -\sqrt{3}i \\ 0 & 0 & \sqrt{3}i & 0 \end{pmatrix}$$

since $\hat{J}^2 |J, m\rangle = J(J+1)\hbar^2 |J, m\rangle$,

$$\hat{J}^2 = \frac{15\hbar^2}{4} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \frac{15\hbar^2}{4} \hat{I}$$

2.

$$\text{since } J_\pm = J_x \pm iJ_y, \Rightarrow \begin{cases} J_x = \frac{J_+ + J_-}{2} \\ J_y = \frac{J_+ - J_-}{2i} \end{cases}$$

$$\langle j, m|J_x|j, m\rangle = \frac{1}{2}(\langle j, m|J_+|j, m\rangle + \langle j, m|J_-|j, m\rangle) = \frac{1}{2}(c_1\langle j, m|j, m+1\rangle + c_2\langle j, m|j, m-1\rangle) = 0$$

Similarly, $\langle j,m|J_y|j,m\rangle = 0$

$$\begin{aligned}
\langle j,m|J_x^2|j,m\rangle &= \frac{1}{4}\langle j,m|(J_+ + J_-)^2|j,m\rangle = \frac{1}{4}\langle j,m|(J_+^2 + J_-^2 + J_+J_- + J_-J_+)|j,m\rangle \\
&= \frac{1}{4}\langle j,m|(J_+J_- + J_-J_+)|j,m\rangle \\
&= \frac{\hbar^2}{4}\{\sqrt{j(j+1)-m(m-1)}\sqrt{j(j+1)-(m-1)(m-1+1)} \\
&\quad +\sqrt{j(j+1)-m(m+1)}\sqrt{j(j+1)-(m+1)(m+1-1)}\} \\
&= \frac{\hbar^2}{4}\{j(j+1)-m(m-1)+j(j+1)-m(m+1)\} \\
&= \frac{\hbar^2}{2}\{j(j+1)-m^2\}
\end{aligned}$$

Similarly, we can get $\langle j,m|J_y^2|j,m\rangle = \frac{\hbar^2}{2}\{j(j+1)-m^2\}$

(3) Using the results in (2), it is true that $\Delta J_x \Delta J_y = \sqrt{\langle J_x^2\rangle\langle J_y^2\rangle} = \frac{\hbar^2}{2}\{j(j+1)-m^2\}$
Meanwhile,

$$\frac{1}{2}|\langle j,m|[J_x,J_y]|j,m\rangle = \frac{\hbar}{2}\langle j,m|J_z|j,m\rangle = \frac{\hbar^2}{2}|m|$$

For $|m| \leq J$, it is always true that $j(j+1)-m^2-|m| \geq 0$. Thus $\Delta J_x \Delta J_y \geq \frac{1}{2}|\langle [J_x,J_y]\rangle|$ and only when $m = \pm j$, the equality survives.

4.

$$\begin{aligned}
U[R(\alpha,\beta,\gamma)] &= e^{-i\alpha \hat{J}_z/\hbar}e^{-i\beta \hat{J}_y/\hbar}e^{-i\gamma \hat{J}_z/\hbar} \\
&= \sum_m \sum_n \sum_{n'} \sum_{m'} e^{-i\alpha \hat{J}_z/\hbar}|z,m\rangle\langle z,m|y,n\rangle\langle y,n|e^{-i\beta \hat{J}_y/\hbar}|y,n'\rangle\langle y,n'|z,m'\rangle\langle z,m'|e^{-i\gamma \hat{J}_z/\hbar} \\
&= \sum_m \sum_n \sum_{m'} e^{-i\alpha m}|z,m\rangle\langle z,m|y,n\rangle e^{-i\beta n}\langle y,n'|z,m'\rangle e^{-i\gamma m'}|z,m\rangle\langle z,m'| \\
&= \begin{pmatrix} e^{-i\alpha} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & e^{i\alpha} \end{pmatrix} T \begin{pmatrix} e^{-i\beta} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & e^{i\beta} \end{pmatrix} T^+ \begin{pmatrix} e^{-i\gamma} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & e^{i\gamma} \end{pmatrix} \\
&= \begin{pmatrix} e^{-i\alpha} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & e^{i\alpha} \end{pmatrix} \begin{pmatrix} \frac{1+cos\beta}{2} & \frac{-sin\beta}{\sqrt{2}} & \frac{1-cos\beta}{2} \\ \frac{sin\beta}{\sqrt{2}} & cos\beta & -\frac{sin\beta}{\sqrt{2}} \\ \frac{1-cos\beta}{2} & \frac{sin\beta}{\sqrt{2}} & \frac{1+cos\beta}{2} \end{pmatrix} \begin{pmatrix} e^{-i\gamma} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & e^{i\gamma} \end{pmatrix}
\end{aligned}$$

Where

$$T = \begin{pmatrix} 1/2 & 1/\sqrt{2} & 1/2 \\ i/\sqrt{2} & 0 & -i/\sqrt{2} \\ -1/2 & 1/\sqrt{2} & -1/2 \end{pmatrix}$$

$$\therefore \Psi = D^{(1)}|1,1\rangle = U\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} \frac{1+cos\beta}{2}e^{-i(\gamma+\alpha)} \\ \frac{sin\beta}{\sqrt{2}}e^{-i\gamma} \\ \frac{1-cos\beta}{2}e^{-i(\gamma-\alpha)} \end{pmatrix}$$

Meanwhile,

$$\begin{aligned}
\overrightarrow{J} &= \hat{J}_x\overrightarrow{e}_x + \hat{J}_y\overrightarrow{e}_y + \hat{J}_z\overrightarrow{e}_z \\
&= \frac{\hbar}{\sqrt{2}}\begin{pmatrix}0 & 1 & 0\\ 1 & 0 & 1\\ 0 & 1 & 0\end{pmatrix}\overrightarrow{e}_x + \frac{\hbar}{\sqrt{2}}\begin{pmatrix}0 & -i & 0\\ i & 0 & -i\\ 0 & -i & 0\end{pmatrix}\overrightarrow{e}_y + \hbar\begin{pmatrix}1 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & -1\end{pmatrix}\overrightarrow{e}_z
\end{aligned}$$

$$\begin{aligned}
\langle\Psi|\overrightarrow{J}_x|\Psi\rangle &= \frac{\hbar}{\sqrt{2}}\left(\ \tfrac{1+cos\beta}{2}e^{i(\gamma+\alpha)},\quad \tfrac{sin\beta}{\sqrt{2}}e^{i\gamma},\quad \tfrac{1-cos\beta}{2}e^{i(\gamma-\alpha)}\ \right)\begin{pmatrix}0 & 1 & 0\\ 1 & 0 & 1\\ 0 & 1 & 0\end{pmatrix}\begin{pmatrix}\tfrac{1+cos\beta}{2}e^{-i(\gamma+\alpha)}\\ \tfrac{sin\beta}{\sqrt{2}}e^{-i\gamma}\\ \tfrac{1-cos\beta}{2}e^{-i(\gamma-\alpha)}\end{pmatrix} \\
&= \frac{\hbar}{4}\{sin\beta(1+cos\beta)e^{i\alpha} + sin\beta(1-cos\beta)e^{-i\alpha} + sin\beta[(1-co\beta)e^{i\alpha} + (1+cos\beta)e^{-i\alpha}]\} \\
&= \hbar sin\beta cos\alpha
\end{aligned}$$

Similarly, we can get $\langle\Psi|J_y|\Psi\rangle = \hbar sin\beta sin\alpha$ and $\langle\Psi|J_z|\Psi\rangle = \hbar cos\beta$. Thus:

$$\langle\Psi|\overrightarrow{J}|\Psi\rangle = \hbar(sin\beta cos\alpha\overrightarrow{e}_x + sin\beta sin\alpha\overrightarrow{e}_y + cos\beta\overrightarrow{e}_z)$$

In order to rotate $|1,1\rangle$ into state $|1,0\rangle$, we should have $1 + cos\beta = 1 - cos\beta = 0$, which is impossible to be satisfied at the same time.Meanwhile,

$$D^{(1)}[R(\alpha,\beta,\gamma)] = \begin{pmatrix}\tfrac{1+cos\beta}{2}e^{-i(\gamma+\alpha)} & \tfrac{-sin\beta}{\sqrt{2}}e^{-i\alpha} & \tfrac{1-cos\beta}{2}e^{i(\gamma-\alpha)}\\ \tfrac{sin\beta}{\sqrt{2}}e^{-i\gamma} & cos\beta & -\tfrac{sin\beta}{\sqrt{2}}e^{i\gamma}\\ \tfrac{1-cos\beta}{2}e^{-i(\gamma-\alpha)} & \tfrac{sin\beta}{\sqrt{2}}e^{i\alpha} & \tfrac{1+cos\beta}{2}e^{i(\gamma+\alpha)}\end{pmatrix}$$

$\langle 1,m|D^{(1)}[R(\alpha,\beta,\gamma)]|1,m'\rangle = D^{(1)}_{mm'}$ and we can always adjust the parameter and make the matrix element non-zero for arbitrary m and m'.

3 III The Angular Momentum states in the Coordinate Basis

$$\begin{aligned}
L_x &= -i\hbar(y\frac{\partial}{\partial z} - z\frac{\partial}{\partial y}) \\
&= -i\hbar[y(\frac{\partial}{\partial r}\frac{\partial r}{\partial z} + \frac{\partial}{\partial\theta}\frac{\partial\theta}{\partial z} + \frac{\partial}{\partial\phi}\frac{\partial\phi}{\partial z}) - z(\frac{\partial}{\partial r}\frac{\partial r}{\partial y} + \frac{\partial}{\partial\theta}\frac{\partial\theta}{\partial y} + \frac{\partial}{\partial\phi}\frac{\partial\phi}{\partial y})] \\
&= -i\hbar[y(\frac{z}{r}\frac{\partial}{\partial r} - \frac{\sqrt{x^2+y^2}}{r^2}\frac{\partial}{\partial\theta}) - z(\frac{y}{r}\frac{\partial}{\partial r} + \frac{zy}{\sqrt{x^2+y^2}r^2}\frac{\partial}{\partial\theta} + \frac{x}{x^2+y^2}\frac{\partial}{\partial\phi})] \\
&= -i\hbar[-(\frac{\sqrt{x^2+y^2}y}{r^2} + \frac{z^2y}{\sqrt{x^2+y^2}r^2})\frac{\partial}{\partial\theta} - \frac{xz}{x^2+y^2}\frac{\partial}{\partial\phi}] \\
&= i\hbar[sin\phi\frac{\partial}{\partial\theta} + ctg\theta cos\phi\frac{\partial}{\partial\phi}]
\end{aligned}$$

Similarly,

$$L_y = i\hbar[-cos\phi\frac{\partial}{\partial\theta} + ctg\theta sin\phi\frac{\partial}{\partial\phi}]; \quad L_z = -i\hbar\frac{\partial}{\partial\phi}$$

$$\begin{aligned}
L_+ = L_x + iL_y &= \hbar[(cos\phi + isin\phi)\frac{\partial}{\partial\theta} + ictg\theta(cos\phi + isin\phi)\frac{\partial}{\partial\phi}] \\
&= \hbar e^{i\phi}(\frac{\partial}{\partial\theta} + ictg\theta\frac{\partial}{\partial\phi}) \\
L_- = L_x - iL_y &= \hbar[(-cos\phi + isin\phi)\frac{\partial}{\partial\theta} - ictg\theta(-cos\phi + isin\phi)\frac{\partial}{\partial\phi}] \\
&= -\hbar e^{-i\phi}(\frac{\partial}{\partial\theta} - ictg\theta\frac{\partial}{\partial\phi})
\end{aligned}$$

$$\begin{aligned}
\hat{L}^2 &= \hat{L}_+\hat{L}_- + L_z^2 - \hbar L_z \\
&= -\hbar^2 e^{i\phi}(\frac{\partial}{\partial\theta} + ictg\theta\frac{\partial}{\partial\phi})e^{-i\phi}(\frac{\partial}{\partial\theta} - ictg\theta\frac{\partial}{\partial\phi}) - \hbar^2\frac{\partial^2}{\partial\phi^2} + i\hbar^2\frac{\partial}{\partial\phi} \\
&= -\hbar^2\left[\frac{\partial^2}{\partial\theta^2} + ctg^2\theta\frac{\partial^2}{\partial\phi^2} + ctg\theta\frac{\partial}{\partial\theta} - ictg^2\theta\frac{\partial}{\partial\phi} + \frac{i}{sin^2\theta}\frac{\partial}{\partial\phi} - i\frac{\partial}{\partial\phi} + \frac{\partial^2}{\partial\phi^2}\right] \\
&= -\hbar^2\left[\frac{\partial^2}{\partial\theta^2} + ctg\theta\frac{\partial}{\partial\theta} + \frac{1}{sin^2\theta}\frac{\partial^2}{\partial\phi^2}\right] \\
&= -\hbar^2\left[\frac{1}{sin\theta}\frac{\partial}{\partial\theta}sin\theta\frac{\partial}{\partial\theta} + \frac{1}{sin^2\theta}\frac{\partial^2}{\partial\phi^2}\right]
\end{aligned}$$

2. Since $Y_l^l(\theta,\phi) = c(sin\theta)^l e^{il\phi}$;

$$\begin{aligned}
\frac{\partial}{\partial\theta}Y_l^l(\theta,\phi) &= cl(sin\theta)^{l-1}cos\theta e^{il\phi} \\
ictg\theta\frac{\partial}{\partial\phi}Y_l^l(\theta,\phi) &= ictg\theta(sin\theta)^l(il)e^{il\phi} = -l(sin\theta)^{l-1}e^{il\phi} \\
\therefore (\frac{\partial}{\partial\theta} &+ ictg\theta\frac{\partial}{\partial\phi})Y_l^l(\theta,\phi) = 0
\end{aligned}$$

Meanwhile, as shown in Part (1), $L_+ = \hbar e^{i\phi}(\frac{\partial}{\partial\theta} + ictg\theta\frac{\partial}{\partial\phi})$, the equation in this problem is just $L_+|l,l\rangle = 0$, which is just the case for state $|l,l\rangle$.

(3) Since $\hat{L}_-|l,m\rangle = \hbar\sqrt{l(l+1) - m(m-1)}|l,m-1\rangle$, we can get the state $|l,l-2\rangle$ from the relation $\hat{L}_-^2|l,l\rangle = \hbar^2\sqrt{(2l)(4l-2)}|l,l-2\rangle$; meanwhile, $\langle\theta\phi|l,l\rangle = Y_l^l(\theta,\phi) = (-1)^l[\frac{(2l+1)!}{4\pi}]^{1/2}\frac{1}{2^l l!}(sin\theta)^l e^{il\theta}$, thus:

$$\begin{aligned}
\langle\theta,\phi|l,l-2\rangle &= \frac{1}{\hbar^2\sqrt{4l(2l-1)}}(-1)^l[\frac{(2l+1)!}{4\pi}]^{1/2}\frac{1}{2^l l!}\hbar^2[e^{-i\phi}(\frac{\partial}{\partial\theta} - ictg\theta\frac{\partial}{\partial\phi})]^2(sin\theta)^l e^{il\theta} \\
&= \frac{1}{\sqrt{4l(2l-1)}}(-1)^l[\frac{(2l+1)!}{4\pi}]^{1/2}\frac{1}{2^l l!}[e^{-i\phi}(\frac{\partial}{\partial\theta} - ictg\theta\frac{\partial}{\partial\phi})][2l(sin\theta)^{l-1}cos\theta e^{i(l-1)\phi}]
\end{aligned}$$

$$= \quad (-1)^{l-2}\sqrt{\frac{l}{2l-1}}[\frac{(2l+1)!}{4\pi}]^{1/2}\frac{1}{2^l l!}[2(l-1)cos^2\theta - sin^2\theta]sin^{l-2}\theta e^{i(l-2)\phi}$$