

**Physics 580: Quantum Mechanics I**  
**Department of Physics, UIUC**  
**Fall Semester 2006**  
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**Problem Set No. 4:**  
**Quantum Mechanical Systems in One Dimension**  
**Due Date: October 16, 2006**

**1 Bound States with Delta Function Potentials**

Consider the problem of a particle of mass  $m$  moving on a line. The particle moves in the presence of the potential  $U(x)$  given by

$$U(x) = g_+ \delta(x - a) + g_- \delta(x + a)$$

where  $a > 0$ ;  $g_+ < 0$  and  $g_- < 0$  are two coupling constants.

1. Consider first the case of a *symmetric* potential in which  $g_+ = g_- \equiv g$ . Find the *bound states* for this case. State clearly what boundary conditions you use and why. How many bound states do you find? Discuss and explain the dependence of your results on the parameters  $g$  and  $a$ . In particular discuss the limits of  $a$  small and large (what is the criterion for this?). What can you say about the symmetry properties of the wave functions of the two bound states. Note: you may use a graphical argument to solve the bound state equations. How similar/different are these wave functions compared to the wave functions of two infinitely separated attractive  $\delta$ -function potentials ?
2. What happens with the bound states discussed above if the two couplings are not equal? Find the wave functions for this general case and discuss what happens to the bound states as one of the couplings, say  $g_-$ , is slowly turned off. How many bound states survive?
3. Solve the Schrödinger Equation for the positive energy states  $E > 0$ , in terms of an expression for the transmitted and reflected amplitudes. Note: you may use any method you wish; the contour integration method is very useful here, but so is the transfer matrix method.
4. Calculate the *transmission* and *reflection* coefficients  $T(E)$  and  $R(E)$ , and the respective phase shifts  $\delta_T(E)$  and  $\delta_R(E)$ , for the case of  $E > 0$ . Are there any resonances?

## 2 The Tunnel Effect

Consider once again a particle of mass  $m$  in one dimension. The particle moves in the potential

$$U(x) = \begin{cases} U_0 > 0, & \text{for } |x| < a \\ 0 & \text{otherwise} \end{cases}$$

1. Derive an expression for the *transmission* and *reflection* coefficients  $T(E)$  and  $R(E)$ , and for the respective phase shifts  $\delta_T(E)$  and  $\delta_R(E)$ , for the energy range  $E > U_0$ . Discuss the behavior of these coefficients and of their associated phase shifts as a function of energy. For what values of  $E$  do you get *perfect transmission*? How do the phase shifts behave near these energies? How do the reflection and transmission coefficients behave for  $E - U_0 \ll U_0$ ? And for  $E \gg U_0$ ? Compare your results with the prediction of Classical Mechanics. For what range is Classical Mechanics compatible with Quantum Mechanics? Why?
2. Consider now the *tunneling regime*  $0 < E < U_0$ . Repeat for this energy range the same analysis you just did for  $E > U_0$ . Is the energy dependence oscillatory or monotonous? Can you get perfect transmission or reflection in this energy range? Find the behavior of the transmission and reflection coefficients when  $U_0 - E \ll U_0$  and when  $E \ll U_0$ . Discuss and explain your results for all of these questions.

## 3 Particle in a One-Dimensional Periodic Potential

In this problem you will consider the problem of a particle of mass  $m$  moving on a line of length  $L \rightarrow \infty$ , whose wave functions obey *periodic boundary conditions* at the endpoints of the line. The particle moves in a *periodic* potential  $U(x)$  of period  $\ell$ ,  $U(x + \ell) = U(x)$ , where for simplicity we have set  $L = N\ell$  and  $N \rightarrow \infty$ .

1. Let  $\hat{T}(\ell)$  be the operator

$$\hat{T}(\ell) = e^{i\frac{\ell}{\hbar}\hat{P}}$$

Recall that this operator translates position states by  $\ell$ . Consider a general Hamiltonian

$$\hat{H} = \frac{\hat{P}^2}{2m} + U(x)$$

Show that if  $U(x)$  is periodic with period  $\ell$ , then

$$[\hat{H}, \hat{T}(\ell)] = 0$$

Show that the eigenvectors of  $\hat{T}(\ell)$  have the form

$$\psi(x) = u_k(x)e^{ikx}$$

where

$$u_k(x + \ell) = u_k(x)$$

This result is known as Bloch's Theorem, and these states are called Bloch waves.

2. Find the allowed values of  $k$  for a system with finite length  $L = Na$ .
3. Consider now the periodic potential  $U(x)$  is

$$U(x) = \sum_{n=-\infty}^{\infty} g\delta(x - n\ell)$$

with  $g > 0$ . This is known as the Kronig-Penney potential. Use Bloch states to find the eigenfunctions for this potential.

4. Find the quantization condition obeyed by the allowed eigenstates for the Kronig-Penney potential. Use a graphical argument to show that the allowed states form bands. How many states are there in each band for a system of length  $L$ ? Show that as  $L \rightarrow \infty$  these states form continuous bands. Show that, in the limit  $L \rightarrow \infty$ , the allowed energies for large  $g$  are restricted to narrow bands which concentrate near certain values  $E_n(\infty)$ . Find these energies, give an estimate of the width of these energy *bands*, and find an expression for the wave functions in this limit. Discuss what happens in the opposite limit,  $g$  small. Show that the eigenstates of the particle almost obey the free particle spectrum, except for a set of discrete values of the energy for which no states are allowed. Find these values. Discuss the form of the wave functions in this limit.

## 4 Quantum States of the Linear Harmonic Oscillator

In this problem you will consider the states of a linear harmonic oscillator (LHO) of mass  $m$  and angular frequency  $\omega$  whose Hamiltonian is

$$\hat{H} = \frac{\hat{P}^2}{2m} + \frac{1}{2}m\omega^2\hat{X}^2$$

where  $\hat{X}$  and  $\hat{P}$  are the quantum mechanical position and momentum operators and obey the commutation relation

$$[\hat{X}, \hat{P}] = i\hbar$$

Let us define the *creation* and *annihilation* operators  $\hat{a}^\dagger$  and  $\hat{a}$ , defined by

$$\hat{a}^\dagger = \left(\frac{m\omega}{2\hbar}\right)^{1/2} \hat{X} - \frac{i}{(2m\omega\hbar)^{1/2}} \hat{P}, \quad \hat{a} = \left(\frac{m\omega}{2\hbar}\right)^{1/2} \hat{X} + \frac{i}{(2m\omega\hbar)^{1/2}} \hat{P}$$

1. Calculate the commutator

$$[\hat{a}, \hat{a}^\dagger]$$

2. Show that

$$\hat{H} = \hbar\omega \left( \hat{a}^\dagger \hat{a} + \frac{1}{2} \right)$$

3. Use this result to show that the ket  $|0\rangle$ , defined by  $\hat{a}|0\rangle = 0$ , is the ground state of the LHO, and that the states,

$$|n\rangle = A_n (\hat{a}^\dagger)^n |0\rangle$$

are the eigenstates of  $\hat{H}$ , with eigenvalues  $E_n = \hbar\omega \left( n + \frac{1}{2} \right)$ , with  $n = 0, 1, 2, \dots$ . Find the normalization constant if the states  $\{|n\rangle\}$  are required to be orthonormal.

4. Use the creation-annihilation algebra to calculate the following matrix elements: a)  $\langle n | \hat{X} | n' \rangle$ , b)  $\langle n | \hat{P} | n' \rangle$ , and c)  $\langle 3 | \hat{X}^3 | 2 \rangle$ .
5. Let  $z = u + iv$  be a complex number. Let us consider the *coherent state*  $|z\rangle$  defined by

$$|z\rangle = e^{z\hat{a}^\dagger} |0\rangle$$

- (a) Show that the coherent state  $|z\rangle$  can be expanded in the oscillator eigenkets  $|n\rangle$  as a power series

$$|z\rangle = \sum_{n=0}^{\infty} \frac{z^n}{\sqrt{n!}} |n\rangle$$

- (b) Show that  $|z\rangle$  is a right eigenket of  $\hat{a}$ ,  $\hat{a}|z\rangle = z|z\rangle$ , and that  $\langle z|$  is a left eigenbra of  $\hat{a}^\dagger$ ,  $\langle z|\hat{a}^\dagger = \langle z|z^*$ , where  $z^* = u - iv$ .
- (c) Use the (Baker-Hausdorff-Campbell) identity

$$e^{\hat{A}} e^{\hat{B}} = e^{\hat{A} + \hat{B}} e^{\frac{1}{2}[\hat{A}, \hat{B}]} = e^{\hat{B}} e^{\hat{A}} e^{[\hat{A}, \hat{B}]}$$

which is correct provided  $[\hat{A}, [\hat{A}, \hat{B}]] = [\hat{B}, [\hat{A}, \hat{B}]] = 0$  to show that the inner product of two coherent states  $|z\rangle$  and  $|w\rangle$  is

$$\langle z | w \rangle = e^{z^* w}$$

- (d) Show that the wave function  $\psi_z(x)$  for the coherent state  $|z\rangle$  is

$$\psi_z(x) = \langle x | z \rangle = \left( \frac{m\omega}{\pi\hbar} \right)^{1/4} e^{-((\text{Re } z)^2)/2} e^{-(m\omega/2\hbar) x^2} e^{\sqrt{2m\omega/\hbar} z x}$$

## 5 Bound States of the Pöschel-Teller Potential

Let us consider a particle of mass  $m$  moving in the Pöschel-Teller potential

$$U(x) = - \frac{U_0}{\cosh^2(\alpha x)}$$

where the energy scale  $U_0 > 0$ :  $\alpha$  has units of  $(\text{length})^{-1}$  and it specifies the range of the potential. In this problem you will have to find the spectrum of bound states, and hence  $E < 0$ . It is useful to define the dimensionless parameters  $\lambda$  and  $s$

$$\lambda = \sqrt{\frac{2m|E|}{\hbar^2 \alpha^2}}, \quad \frac{2mU_0}{\alpha^2 \hbar^2} = s(s+1)$$

and to change variables to

$$u = \frac{1}{2}(1 - \tanh(\alpha x))$$

where  $0 < u < 1$  for  $-\infty < x < \infty$ . We will write the wave function  $\psi(x)$  as

$$\psi(x) = (4u(1-u))^{\lambda/2} w(u)$$

Below you will find that the solutions can be written in terms of the hypergeometric function  $F(\alpha, \beta, \gamma, u)$  (defined below)

1. Find the asymptotic form of the wave functions  $\psi(x)$  for  $E < 0$  for  $|x| \rightarrow \infty$  and for  $x \rightarrow 0$ . What do these asymptotic behaviors imply for the behavior of the function  $w(u)$  as  $u \rightarrow 0$  and  $u \rightarrow 1$ ?
2. Show that the function  $w(u)$  is a solution of the hypergeometric equation

$$u(1-u) \frac{d^2 w}{du^2} + [\gamma - (\alpha + \beta + 1)u] \frac{dw}{du} - \alpha\beta w = 0$$

What is the relation between  $\alpha, \beta, \gamma$  and the parameters  $\lambda$  and  $s$ ?

3. Is there a solution of the hypergeometric equation which satisfies the boundary conditions required for  $w(u)$  so that  $\psi(x)$  is in the physical Hilbert space? Show that the boundary conditions can only be satisfied if the series of the hypergeometric function terminates and this function thus becomes a polynomial of finite degree  $n$ . Find the allowed values of  $n$  and what restrictions do they imply to the possible values of  $\lambda$  and  $s$ . Recall the discussion we went through in class for the linear harmonic oscillator.
4. Use the results you just derived to find a formula for the allowed energy levels. How many levels do you find? What parameter(s) control the number of allowed bound states?

Useful properties of the hypergeometric function:

The hypergeometric equation has two linearly independent solutions  $F(\alpha, \beta, \gamma, u)$  and  $u^{1-\gamma}F(\beta - \gamma + 1, \alpha - \gamma + 1, 2 - \gamma, u)$ . The hypergeometric function  $F(\alpha, \beta, \gamma, u)$  is an analytic function of the (complex) variable  $u$ , for  $|u| < 1$  where it has the power series expansion

$$F(\alpha, \beta, \gamma, u) = 1 + \frac{\alpha\beta}{\gamma} \frac{u}{1!} + \frac{\alpha(\alpha+1)\beta(\beta+1)}{\gamma(\gamma+1)} \frac{u^2}{2!} + \dots$$

where  $\gamma \neq 0, -1, -2, \dots$ . The hypergeometric function also satisfies, among several others, the following useful identity

$$\begin{aligned} F(\alpha, \beta, \gamma, u) &= \frac{\Gamma(\gamma)\Gamma(\gamma - \alpha - \beta)}{\Gamma(\gamma - \alpha)\Gamma(\gamma - \beta)} F(\alpha, \beta, \alpha + \beta + 1 - \gamma, 1 - u) \\ &\quad + \frac{\Gamma(\gamma)\Gamma(\alpha + \beta - \gamma)}{\Gamma(\alpha)\Gamma(\beta)} (1 - u)^{\gamma - \alpha - \beta} F(\gamma - \alpha, \gamma - \beta, \gamma + 1, 1 - u) \end{aligned}$$