Throught these solutions I will make extensive usage of the natural system of units, in which a set of three parameters of the problem is set to unity. By dimensional analysis one can recover the original units. The choice of parameters depends on the problem but in many cases they are $\hbar$ the mass and some distance. Useful for the dimensional analysis is to remember the following:

1. The 1-d wave function has dimension of $[L^{-1/2}]$ because is satisfies $\int dx\, |\psi(x)|^2 = 1$ which is dimensionless
2. The delta function has dimension of $[L^{-1}]$ because $\int dx \delta(x) = 1$
3. $[\hbar] = [Energy][T] = [ML^2T^{-1}]$
4. $[Energy] = [\hbar^2][M^{-1}L^{-2}]$
5. The argument of any analytic function that is not a simple power must be dimensionless.

In the above, L is for length, M for mass and T for time.

To restore the original units of some quantity Q, one has to find the dimensionality of this quantity (say in SI units) and then express it with respect to the parameters that were set to 1. For example lets say that

$$\hbar = m = a = 1$$

and we have an expression involving g the delta function depth. The unit of energy is $\frac{\hbar^2}{ma^2}$ and $g[L^{-1}]$ where $[L^{-1}]$ comes from the delta function, has the units of energy. This means that the units for g is $\frac{\hbar^2}{ma}$. So after we perform our calculation and end up with an expression involving g, to restore the original units we replace g by $\frac{mag}{\hbar^2}$. Similarly the unit for the wave vector k is obviously a^{-1} and restoring the units would give ka.

1 Bound States with Delta function potentials

1) and 2)

It is more efficient for this problem to first treat the general problem with g_- and g_+ not necessarily equal. After I do this I will adopt my solution for parts 1 and 2. To simplify notation I will use natural system of units:

$$\hbar = m = a = 1$$

This is equivalent to introducing dimensionless quantities:

$$\begin{aligned} k &= k'/a \\ g &= g' m a \hbar^{-2} \end{aligned}$$

The boundary conditions for the wave function are the following:

1. The wave function for a bound state must vanish as $|x| \to \infty$
2. The wave function is continuous everywhere
3. the derivative is continuous everywhere unless the potential has delta peaks of height g, say at some position a. In this case by integrating the Schrodinger equation from a point slightly before the peak to slightly after it we get
$$\psi'(a+0^+) - \psi'(a-0^+) = 2g\psi(a)$$

For any x apart from the peak position the wave function satisfies the free particle Schrodinger equation. Since we are looking for bound states the wave vector must be an imaginary number (ie the wave function is damped). A possible wave function which vanishes at infinity is:

$$\psi(x) = \begin{cases} A_1 e^{k(x+1)} & x < -1 \\ Ae^{k(x-1)} + Be^{-k(x+1)} & |x| < 1 \\ A_3 e^{-k(x-1)} & x > 1 \end{cases}$$

where $k = \sqrt{2mE}$.
Imposing continuity in $x = \pm a$ gives the conditions:

$$\begin{aligned} A_1 &= Ae^{-2k} + B \\ A_3 &= A + Be^{-2k} \end{aligned}$$

It is very rewarding if we just plug in these results in our wave function because we just get:

$$\psi(x) = Ae^{-k|x-1|} + Be^{-k|x+1|} \tag{1}$$

The algebra can be really nasty if we don't start from this form, as many of the students have realized.

Now we will apply the continuity of the derivative of the wave function at $x = \pm a$. In this form of the wave function this is easy because the change of the derivative of $e^{-k|x\pm a|}$ is just $-2k$:

$$\begin{aligned} -2kB &= 2g_- \left(Ae^{-2k} + B\right) \\ -2kA &= 2g_+ \left(A + Be^{-2k}\right) \end{aligned}$$

Because $g_+, g_- < 0$ we can write this system of equations as:

$$Ae^{-2k} + \left(1 + \frac{k}{g_-}\right) B = 0 \tag{2}$$

$$\left(1 + \frac{k}{g_+}\right) A + Be^{-2k} = 0 \tag{3}$$

If we bring the B terms in the right side and then multiply these equations we will get:

$$\frac{A^2}{B^2} = \frac{1 + \frac{k}{g_-}}{1 + \frac{k}{g_+}}$$

This form implies that the denominator and the numerator must always have the same sign. The most general solution of this equation is:

$$\begin{aligned} A &= C\sqrt{\left|1+\frac{k}{g_-}\right|} \\ B &= \epsilon C\sqrt{\left|1+\frac{k}{g_+}\right|} \end{aligned}$$

where $\epsilon = \pm 1$ and we already see here that there may be two bound states. Lets write down the final form of the wave function:

$$\psi(x) = C\sqrt{\left|1+\frac{k}{g_-}\right|}e^{-k|x-1|} + \epsilon C\sqrt{\left|1+\frac{k}{g_-}\right|}e^{-k|x+1|}$$

The normalization can be obtained by:

$$1 = 4AB\left(1+\frac{e^{-2k}}{2k}\right) + \frac{(A-B)^2}{2k}$$

Not both A and B can be zero so the determinant of this system must be zero. This gives us the quantization condition:

$$e^{-4k} = \left(1+\frac{k}{g_+}\right)\left(1+\frac{k}{g_-}\right)$$

or after restoring the original units:

$$e^{-4ka} = \left(1+\frac{\hbar^2 k}{m g_+}\right)\left(1+\frac{\hbar^2 k}{m g_-}\right)$$

The right hand side is a parabola which starts from 1 at the origin and changes sign at $k = -g_-$ and $k = -g_+$. Without any loss of generality we will assume that

$$|g_-| < |g_+|$$

If this is not the case we can make a mirror transformation and a relabeling. Then the right side will change sign once at $k = -g_-$ and again at $k = -g_+$. When $k = -g_- - g_+$ it will become 1 again. Since the left side is between 0 and 1 we care to search for solutions in the following intervals:

1. $0 \le k \le -g_-$. Both sides are starting from 1 and decrease. The right side reaches zero and the left $e^{4g_-} > 0$. In order to have a non trivial solution here the derivative of the left hand side at zero must be lower than of the right hand side. This gives us the condition:

$$4 > -g_-^{-1} - g_+^{-1} \tag{4}$$

Also in this regime both $1 + \frac{k}{g_-}$ and $1 + \frac{k}{g_+}$ are positive and A and B have opposite sings (because of 2). This corresponds to $\epsilon = -1$. Note that since the energy is $-\frac{k^2}{2}$ this is actually the most energetic. Therefore it is an excited state and the wave function has a bond. In the limit of the condition the right hand side is tangential to the left hand side and the only solution is $k = 0$. This means that for too small $g_\pm$ the excited solution will fall into the continuous spectrum.

2. $-g_+ \leq k \leq -g_+ - g_-$. Here the right hand side is increasing starting from zero up to 1 but the left hand side is decreasing and is less than 1. There is always at least one solution in this regime. Also both $1 + \frac{k}{g_-}$ and $1 + \frac{k}{g_+}$ are negative and therefore A and B have the same signs. This corresponds to $\epsilon = +1$. This corresponds to the low energy ground state of the system.

If we start turning off one of the two potentials we will have two bound states until Esq. 4 is not satisfied any more. During the transition A/B will grow which means that the wave function will have more weight around the $x = a$ delta potential.

Now consider the case $g_- = g_+ = g$. Now $A = \epsilon B$ and the wave function can be written as:

$$\psi(x) = Ae^{-k|x-a|} + \epsilon Ae^{-k|x+a|} \tag{5}$$

We can see that for $\epsilon = 1$ $\psi(x)$ is symmetric and for $\epsilon = -1$ it is antisymmetric so ϵ indicates the parity. The reason that the wave function is of a particular symmetry is that the Hamiltonian is symmetric and therefore its eigenfunction must be also eigenfunction of parity.

After I restore the original units, the quantization condition reads:

$$e^{-4ka} = (1 + \frac{\hbar^2 k}{mg})^2$$

There is always one bound state in the range $1 \leq \frac{\hbar^2 k}{m|g|} \leq 2$ and in order to have a bound state in the range $0 \leq \frac{\hbar^2 k}{m|g|} \leq 1$ the following condition must be satisfied:

$$\frac{1}{2} < \frac{ma\,|g|}{\hbar^2}$$

Notice that the parameter $a\,|g|$ is the one determining the number of bound states. Too small g (weak potential) or too small a (too close together) will make the antisymmetric excited wave function disappear as discussed before.

We recall that for a single delta potential at $x = 0$ the wave function is $e^{-k|x|}$ where $k = -gm\hbar^{-2}$. For large enough ga we have two bound states and the wave function of eq.. 1 certainly has the form of a symmetric or antisymmetric sum of two single-delta potential wave function. However to say so the value of k must be right. We notice that for fixed g but extremely small a the left hand side of the quantization condition vanishes which gives the right value of k. On the other hand for very small a we can ignore the ka terms in the exponents in the wave function which will give something vanishing for $\epsilon = -1$ and for $\epsilon = 1$ it will give the wave function of a single delta. Also the left side of the quantization condition will be one and there will be two solutions one with $k = 0$ (for $\epsilon = -1$) which is not a bound state and another with $k = -(2g)m\hbar^{-2}$ which corresponds to the wave function of a delta potential with twice the height.

3)

We will use contour integration method. First we consider an incoming wave function of the form:

$$\psi_i(x) = e^{i\lambda x}$$

with $\lambda = \sqrt{2E}$. Schrodinger equation can be written as:

$$\left(E + \frac{1}{2}\frac{\partial^2}{\partial x^2}\right)\psi(x) = U(x) = g_-\psi(-a)\delta(x+a) + g_+\psi(a)\delta(x-a)$$

In Fourier space this equation can be written as:

$$\left(E - \frac{k^2}{2}\right)\psi(k) = g_-\psi(-a)e^{ika} + g_+\psi(a)e^{-ika}$$

The general solution of the Schrodinger equation is:

$$\begin{aligned}
\psi(x) &= \psi_i(x) + \int \frac{dk}{2\pi} e^{ikx}\psi(k) \\
&= e^{i\lambda x} + g_-\psi(-a)\int \frac{dk}{2\pi}\frac{e^{ik(x+a)}}{E - \frac{k^2}{2} + i\epsilon} + g_+\psi(a)\int \frac{dk}{2\pi}\frac{e^{ik(x-a)}}{E - \frac{k^2}{2} + i\epsilon}
\end{aligned}$$

In the lecture notes it was shown that:

$$\int \frac{dk}{2\pi}\frac{e^{ik(x-a)}}{E - \frac{k^2}{2} + i\epsilon} = \frac{1}{i\lambda}e^{i\lambda|x|}$$

Using this directly gives:

$$\psi(x) = e^{i\lambda x} + \psi(-a)\frac{g_-}{i\lambda}e^{i\lambda|x+a|} + \psi(a)\frac{g_+}{i\lambda}e^{i\lambda|x-a|}$$

Now we impose the conditions:

$$\begin{aligned}
\psi(-a) &= e^{-i\lambda a} + \frac{g_-}{i\lambda}\psi(-a) + \psi(a)\frac{g_+}{i\lambda}e^{2ia\lambda} \\
\psi(a) &= e^{i\lambda a} + \psi(-a)\frac{g_-}{i\lambda}e^{2i\lambda a} + \frac{g_+}{i\lambda}\psi(a)
\end{aligned}$$

We can solve this system of equations for $\psi(-a)$ and $\psi(a)$. We write the system in matrix form:

$$\begin{pmatrix} 1 + i\frac{g_+}{\lambda} & i\frac{g_-}{\lambda}e^{2ia\lambda} \\ i\frac{g_+}{\lambda}e^{2i\lambda a} & 1 + i\frac{g_-}{\lambda} \end{pmatrix}\begin{pmatrix} \psi(a) \\ \psi(-a) \end{pmatrix} = \begin{pmatrix} e^{i\lambda a} \\ e^{-i\lambda a} \end{pmatrix}$$

The determinant of the matrix is:

$$\begin{aligned} D &= \left(1+i\frac{g_+}{\lambda}\right)\left(1+i\frac{g_-}{\lambda}\right)+\frac{g_+}{\lambda}\frac{g_-}{\lambda}e^{4ia\lambda} \\ &= 1-\frac{g_+g_-}{\lambda^2}(1-\cos(4\lambda a))+i\left[\frac{g_++g_-}{\lambda}+\frac{g_+g_-}{\lambda^2}\sin(4\lambda a)\right] \end{aligned}$$

The solution is

$$\begin{pmatrix} \psi(a) \\ \psi(-a) \end{pmatrix} = \frac{1}{D}\begin{pmatrix} 1+i\frac{g_-}{\lambda} & -i\frac{g_-}{\lambda}e^{2ia\lambda} \\ -i\frac{g_+}{\lambda}e^{2i\lambda a} & 1+i\frac{g_+}{\lambda} \end{pmatrix}\begin{pmatrix} e^{i\lambda a} \\ e^{-i\lambda a} \end{pmatrix}$$

which eventually gives:

$$\begin{aligned} D\psi(a) &= e^{i\lambda a} \\ D\psi(-a) &= e^{-i\lambda a}+2\frac{g_+}{\lambda}e^{+i\lambda a}\sin(2\lambda a) \end{aligned}$$

The solution for $x>a$ is the transmitted wave $\psi_T(x)=Ae^{i\lambda x}$ then $A=e^{-i\lambda a}\psi_T(a)=e^{-i\lambda a}\psi(a)$ and the transmitted amplitude is:

$$DA = 1$$

The reflected amplitude is a bit trickier because it is not the whole wave function at $x<-a$ but only the $e^{-i\lambda x}$ component. For $x<-a$ the wave function can be written as:

$$\psi(x)-e^{i\lambda x}=\psi(-a)\frac{g_-}{i\lambda}e^{-i\lambda a}e^{-i\lambda x}+\psi(a)\frac{g_+}{i\lambda}e^{i\lambda a}e^{i\lambda x}=Be^{-i\lambda x}$$

where the reflected amplitude B is given by:

$$\begin{aligned} DB &= D\psi(-a)\frac{g_-}{i\lambda}e^{-i\lambda a}+D\psi(a)\frac{g_+}{i\lambda}e^{i\lambda a} \\ &= \left(e^{-i\lambda a}+2\frac{g_+}{\lambda}e^{+i\lambda a}\sin(2\lambda a)\right)\frac{g_-}{i\lambda}e^{-i\lambda a}+\frac{g_+}{i\lambda}e^{2i\lambda a} \\ &= \left(\frac{g_+}{\lambda}-\frac{g_-}{\lambda}-2i\frac{g_+}{\lambda}\frac{g_-}{\lambda}\right)\sin(2\lambda a)-i\left(\frac{g_+}{\lambda}+\frac{g_-}{\lambda}\right)\cos(2\lambda a) \end{aligned}$$

Note that for $\lambda\to\infty$ $D\to 1$ and $B\to 0$. In this case there is perfect transmission.

4)

One can show that the amplitudes satisfy the conservation of particle number thus:

$$|DA|^2+|DB|^2=1$$

First we deal with D

$$\begin{aligned} |D|^2 &= \left[1-\frac{g_+g_-}{\lambda^2}(1-\cos(4\lambda a))\right]^2+\left[\frac{g_++g_-}{\lambda}+\frac{g_+g_-}{\lambda^2}\sin(4\lambda a)\right]^2 \\ &= 1+\frac{g_+^2}{\lambda^2}+\frac{g_-^2}{\lambda^2}+2\frac{g_+^2}{\lambda^2}\frac{g_-^2}{\lambda^2} \\ &\quad +2\frac{g_+}{\lambda}\frac{g_-}{\lambda}\left[\left(1-\frac{g_+}{\lambda}\frac{g_-}{\lambda}\right)\cos(4\lambda a)+\left(\frac{g_+}{\lambda}+\frac{g_-}{\lambda}\right)\sin(4\lambda a)\right] \end{aligned}$$

The argument is:

$$\arg D = \arctan \frac{\frac{\lambda}{g_+} + \frac{\lambda}{g_-} + \sin(4\lambda a)}{\frac{\lambda^2}{g_+ g_-} - 1 + \cos(4\lambda a))}$$

The transmission coefficient and the phase shift are:

$$T(E) = |A|^2 = \frac{1}{|D|^2}$$

$$\delta_T(E) = -\arg D$$

The reflection coefficient and the phase shift are:

$$\begin{aligned} R(E) &= 1 - T(E) = 1 - \frac{1}{|D|^2} \\ \delta_R(E) &= \arg B = \pi + \arctan \frac{2 + \left(\frac{\lambda}{g_+} + \frac{\lambda}{g_-}\right) \cot(2\lambda a)}{\left(\frac{\lambda}{g_+} - \frac{\lambda}{g_-}\right)} - \arg D \end{aligned}$$

The question about the resonances is not as trivial as it seems. The resonances are determined by the behaviour of the determinant D. More specifically close enough to a resonance with energy E_r $D \approx E - E_r + i\Gamma$ where Γ is the width of the resonance. The transition amplitude is proportional to $|D|^{-2}$ and around a resonance it can be approximated by a Lorentzian. From the for of D it is clear that we are looking for the characteristic energies that make the real part of D zero. Then the imaginary part at this energy will give the width Γ. The equation we need to solve is thus:

$$\Re D = 0 = \frac{g_+ g_-}{\lambda^2} \left[\cos(4\lambda) - 1 + \frac{\lambda^2}{g_+ g_-}\right]$$

or:

$$\cos(4\lambda) = 1 - \frac{\lambda^2}{g_+ g_-}$$

The existence of the resonances is controlled only by the parameter $g_+ g_- > 0$. This equation can be solved graphically. The right hand side is a parabola. If $g_+ g_- < 2$ the right side falls faster than the cosine and there is only one trivial solution. Right after $g_+ g_- > 2$ a second solution occurs. As $g_+ g_-$increases the parabola will become tangential to the valley of $\cos(2\lambda)$ around $4\lambda = 2\pi + \pi$ which will give one extra resonance which will split in two slightly after this occurs. Pairs of resonances with increasing energies are added every time the parabola becomes tangential to some valley of the cosine. The number of resonances goes to infinity as $g_+ g_-$ increases.

2 The Tunnel Effect

We consider three regions ($x < -a$, $|x| < a$ and $x > a$) in which the particle satisfies the free particle Schrodinger equation and we also assume that the particles are hitting the barrier from the left, so that there is no left traveling component in the wave function (e^{-ikx}) in the right side. The wave function can be written then as:

$$\psi(x) = \begin{cases} A_1 e^{ikx} + B_1 e^{-ikx} & x < -a \\ A_2 e^{i\lambda x} + B_2 e^{-i\lambda x} & |x| < a \\ A_3 e^{ikx} & x > a \end{cases}$$

where $k = \hbar^{-1}\sqrt{2mE}$ and $\lambda = \hbar^{-1}\sqrt{2m(E - U_0)}$. Notice that this is actually the general solution for any possible energy since if $E < U_0$ λ will be imaginary.

Continuity of the wave function and its derivative at $x = a$ results in the equations

$$\begin{aligned} A_3 e^{ika} &= A_2 e^{i\lambda a} + B_2 e^{-i\lambda a} \\ \frac{k}{\lambda} A_3 e^{ika} &= A_2 e^{i\lambda a} - B_2 e^{-i\lambda a} \end{aligned}$$

We can solve this system of equations easily if we remember that:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \frac{1}{det} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

The solution is:

$$\begin{aligned} A_2 &= \frac{A_3}{2} e^{i(k-\lambda)a} (1 + \frac{k}{\lambda}) \\ B_2 &= \frac{A_3}{2} e^{i(k+\lambda)a} (1 - \frac{k}{\lambda}) \end{aligned}$$

Similarly for $x = -a$:

$$\begin{aligned} A_1 e^{-ika} + B_1 e^{ika} &= A_2 e^{-i\lambda a} + B_2 e^{i\lambda a} \\ &= A_3 e^{ika} \left(\cos(2\lambda a) - i\frac{k}{\lambda} \sin(2\lambda a) \right) \\ A_1 e^{-ika} - B_1 e^{ika} &= \frac{\lambda}{k} A_2 e^{-i\lambda a} - \frac{\lambda}{k} B_2 e^{i\lambda a} \\ &= A_3 e^{ika} \left(\cos(2\lambda a) - i\frac{\lambda}{k} \sin(2\lambda a) \right) \end{aligned}$$

Using the same inversion we can solve for A_1 and B_1 and then express them in terms of A_3:

$$\begin{aligned} A_1 &= A_3 e^{2ika} \left(\cos(2\lambda a) + i \cosh p \sin(2\lambda a) \right) \\ B_1 &= i A_3 \sinh p \sin(2\lambda a) \end{aligned}$$

where we defined:

$$e^p = \frac{\lambda}{k} \Rightarrow p = \frac{1}{2} \ln \left(1 - \frac{U_0}{E}\right)$$

so that

$$\begin{aligned} \sinh p &= -\frac{U_0}{2\sqrt{E(E-U_0)}} \\ \cosh p &= \frac{2E-U_0}{2\sqrt{E(E-U_0)}} \end{aligned}$$

Now we can evaluate the transmission and reflection coefficients:

$$\begin{aligned} T(E) &= \left|A_3 A_1^{-1}\right|^2 = \left(\cos^2(2\lambda a) + \cosh^2 p \sin^2(2\lambda a)\right)^{-1} \\ &= \left(1 + \sinh^2 p \sin^2(2\lambda a)\right)^{-1} \\ R(E) &= \left|B_1 A_1^{-1}\right|^2 = \left|B_1 A_3^{-1}\right|^2 \left|A_3 A_1^{-1}\right|^2 = T(E) \sinh^2 p \sin^2(2\lambda a) \end{aligned}$$

These equations can also be written:

$$\begin{aligned} R(E) + T(E) &= 1 \\ R(E) &= T(E) \frac{U_0^2}{4E(E-U_0)} \sin^2(2\lambda a) \end{aligned}$$

where the first equation is a direct consequence of particle conservation. The phase shift $\delta_T(E)$ of the out going wave is the argument of $\frac{A_3}{A_1}$

$$\begin{aligned} \delta_T(E) &= -\arg\left(e^{2ika}\left(\cos(2\lambda a) + i \cosh p \sin(2\lambda a)\right)\right) \\ &= -2ka + \arctan\left(\cosh p \tan(2\lambda a)\right) \\ &= -2ka + \arctan\left(\frac{2E-U_0}{2\sqrt{E(E-U_0)}} \tan(2\lambda a)\right) \end{aligned}$$

Also the reflected phase shift:

$$\delta_R(E) = -2ka + \arctan\left(\frac{2\sqrt{E(E-U_0)}}{2E-U_0} \cot(2\lambda a)\right)$$

For a classical particle there is no reflection: a classical particle with energy larger than the peak potential energy, would just slow down as it traverses the potential barrier but it would keep moving forward. Therefore $R_{classical}(E) = 0$ for $E > U_0$. Quantum mechanics predicts that there is a probability that the particle actually bounces off the potential. However for large energies $E \gg U_0$ the reflection coefficient vanishes like E^{-2}, so the high energy regime is the classical regime. On the other hand if E is really close to the peak of the potential $E - U_0 \ll U_0$, $\sin^2(2\lambda a) \approx 4\lambda^2 a^2 = 8ma^2\hbar^{-2}(E - U_0)$ and

$T(E) = \left(1 + 2U_0 ma^2\hbar^{-2}\right)^{-1}$. We see also that the term $\sin^2(2\lambda a)$ will induce fluctuations in the reflection and transmission coefficient as a function of $\sqrt{E - U_0}$. These fluctuations are a purely quantum mechanical phenomenon. In the special case when $\sin(2\lambda a) = 0 \Rightarrow n\frac{2\pi}{\lambda} = 4a$ we have perfect transmission like in classical mechanics. This corresponds to energy:

$$E = U_0 + \frac{\pi^2\hbar^2}{8ma^2}n^2$$

The phase shift for transmission is $\delta_T(E) = -2ka = \sqrt{\frac{8ma^2U_0}{\hbar^2} + \pi^2n^2}$ which for large energies is just πn. In this case the reflection phase shift is not defined.

In this case the length $2a$ of the potential region can fit an integer multiple of half DE-Brogli wavelengths $\frac{\pi}{\lambda}$ (λ is the wave vector, sorry for the confusion).

2)

In case $E < U_0$ the intermediate region is classically forbidden. In this case a classical particle would just bounce off. So in this case $R_{classical} = 1$ for any energy in this range. However a quantum mechanical particle has a small probability of going through. To study the potential in this region consider imaginary $\lambda = i|\lambda|$ where $|\lambda| = \hbar^{-1}\sqrt{2m(U_0 - E)}$. The coefficient $2\sinh p = \frac{\lambda}{k} - \frac{k}{\lambda}$ will remain the same but the term $\sin^2(2\lambda a) \to -\sinh^2(2|\lambda|a)$. The reflection and transmission coefficient in this regime become:

$$\begin{aligned} T(E) &= R(E)\frac{4E(U_0 - E)}{U_0^2}\sinh^{-2}(2\lambda a) = \left(1 + \frac{U_0^2}{4E(U_0 - E)}\sinh^2(2\lambda a)\right)^{-1} \\ R(E) &= 1 - T(E) \end{aligned}$$

The term $\sinh^2(2\lambda a)$ may be small but not zero and therefore there is a small probability that the particle will tunnel through the obstacle. Notice here that there are no fluctuating terms in this case and there is no way to get perfect transmission or reflection. Now consider the limit $U_0 - E \ll U_0$ then λ is small, $\sinh^2(2\lambda a) \approx 4\lambda^2a^2 = 8m\hbar^{-2}(U_0 - E)a^2$ and $T(E) = \left(1 + 2U_0ma^2\hbar^{-2}\right)^{-1}$, which is the same as in the previous part for the same regime. In the other limit $E \ll U_0$ we get $T(E) = \frac{4E}{U_0}\sinh^{-2}(2\lambda_0 a)$ with $\lambda_0 = \hbar^{-1}\sqrt{2mU_0}$. In the case that $\lambda_0 a \gg 1$ this can be approximated to: $T(E) = \frac{4E}{U_0}e^{-4\lambda_0 a}$.

3 Particle in a One-Dimensional periodic potential

1)

Consider an arbitrary function $\psi(x)$. Then $\hat{U}(x)\psi(x) = U(x)\psi(x)$ and the following is true:

$$\hat{T}(l)U(x)\psi(x) = U(x + l)\psi(x + l)$$

$$\begin{aligned} &= U(x)\psi(x+l) \\ &= U(x)\hat{T}(l)\psi(x) \end{aligned}$$

Since $\psi(x)$ is arbitrary, $\left[\hat{T}(l), \hat{U}(x)\right] = 0$ and since the translation operator commutes with the momentum $\left[\hat{T}(l), H\right] = 0$.

The eigen functions of $\hat{T}(l)$ satisfy:

$$\hat{T}(l)\psi(x) = \psi(x+l) = z\psi(x)$$

If we have periodic boundary conditions then for some large $L = Nl$:

$$\psi(x+L) = z^N\psi(x) = \psi(x)$$

This means that z must be a root of unity and with no loss of generality we can write it as:

$$z = e^{ikl}$$

Lets write $x = nl + [x]$ where $0 \leq [x] < l$ is the modulo of the division of x with l. Then:

$$\psi(x) = z^n\psi([x]) = e^{ik(nl)}\psi([x]) = e^{ikx}e^{-ik[x]}\psi([x]) = e^{ikx}u_k(x) \tag{6}$$

Because $[x]$ is a periodic function, so is $u_k([x]) = e^{-ik[x]}\psi([x])$.

2)

In this case $a = l$. and $e^{ikNa} = 1$ so that:

$$k = \frac{2\pi}{aN}m$$

for any integer (positive or negative) m.

3)

Again the natural system of units is here to help us with the notation:

$$\hbar = m = l = 1$$

Since the translation operator $\hat{T}(l)$ commutes with the Hamiltonian we can search for eigenstates $\psi_k(x)$ that are also Bloch states (that is they are eigenfunctions of both operators).

It suffices to solve this equation in the interval $S = [-\frac{1}{2}, \frac{1}{2}]$ because the values of the wave function everywhere else can be determined by its values in S. We have two boundary conditions:

1. Continuity of the wave function implies

$$\psi_k(\frac{1}{2} - 0^+) = \psi_k(\frac{1}{2} + 0^+) = e^{ik}\psi_k(-\frac{1}{2})$$

2. Continuity of the derivative of the wave function (everywhere but the location of the peaks) implies that:

$$\psi_k'(\frac{1}{2}-0^+) = \psi_k'(\frac{1}{2}+0^+) = e^{ik}\psi_k(-\frac{1}{2}+0^-)$$

For $x \neq 0$ we have the solution:

$$\psi(x) = \begin{cases} \psi_I(x) = A_I\cos(\lambda x) + B_I\sin(\lambda x) & x<0 \\ \psi_{II}(x) = A_{II}\cos(\lambda x) + B_{II}\sin(\lambda x) & x>0 \end{cases}$$

where $\lambda = \sqrt{2E}$. The continuity of ψ at $x=0$ makes

$$A_I = A_{II} = \frac{A}{2} \tag{7}$$

. The delta peak results in a discontinuity of the derivative

$$\frac{1}{2}\left(\psi_{II}(0^+) - \psi_I(0^+)\right) = g\psi(0) \Rightarrow \lambda(B_{II}-B_I) = g\frac{A}{2}$$

We can express this as:

$$B_{II} = \frac{B}{2} + \frac{g}{\lambda}\frac{A}{2} \tag{8}$$

$$B_I = \frac{B}{2} - \frac{g}{\lambda}\frac{A}{2} \tag{9}$$

Substituting in the wave function we get:

$$\psi(x) = \frac{A}{2}\left(\cos(\lambda x) + \frac{g}{\lambda}\sin(\lambda|x|)\right) + \frac{B}{2}\sin(\lambda x)$$

Also from the boundary conditions we get:

$$\begin{aligned} \psi_{II}(\frac{1}{2}) &= e^{ik}\psi_I(-\frac{1}{2}) \Rightarrow \\ A_{II}\cos(\frac{\lambda}{2}) + B_{II}\sin(\frac{\lambda}{2}) &= e^{ik}A_I\cos(\frac{\lambda}{2}) - e^{ik}B_I\sin(\frac{\lambda}{2}) \\ \psi_{II}'(\frac{1}{2}) &= e^{ik}\psi_I'(-\frac{1}{2}) \Rightarrow \\ -A_{II}\sin(\frac{\lambda}{2}) + B_{II}\cos(\frac{\lambda}{2}) &= e^{ik}A_I\sin(\frac{\lambda}{2}) + e^{ik}B_I\cos(\frac{\lambda}{2}) \end{aligned}$$

These two equation with the help of 7, 8 and 9 can be written as:

$$\begin{aligned} iA\left[\cos\frac{\lambda}{2}\sin\frac{k}{2} + \frac{g}{\lambda}\sin\frac{\lambda}{2}\sin\frac{k}{2}\right] - B\sin\frac{\lambda}{2}\cos\frac{k}{2} &= 0 \\ A\left[\sin\frac{\lambda}{2}\cos\frac{k}{2} - \frac{g}{\lambda}\cos\frac{\lambda}{2}\cos\frac{k}{2}\right] + iB\cos\frac{\lambda}{2}\sin\frac{k}{2} &= 0 \end{aligned}$$

Dividing both sides of each equation with the corresponding coefficient of B gives:

$$iA\left[\cot\frac{\lambda}{2}+\frac{g}{\lambda}\right]\tan\frac{k}{2} \quad = \quad B \tag{10}$$

$$iA\left[\tan\frac{\lambda}{2}-\frac{g}{\lambda}\right]\cot\frac{k}{2} \quad = \quad B \tag{11}$$

The only way for this system of equation to not have a trivial solution $A = B = 0$ is when the determinant is zero. In other words one of the coefficients should be left undetermined because there is not way that the normalization constant can be determined (the Schrodinger equation is linear). Making the determinant zero for this set of equations is like equating their right hand sides, so that their ratio is one:

$$\tan^2\frac{k}{2}=\frac{\tan\frac{\lambda}{2}-\frac{g}{\lambda}}{\cot\frac{\lambda}{2}+\frac{g}{\lambda}}=\frac{1-\cos k}{1+\cos k}$$

where we used

$$\cos k = 2\cos^2\frac{k}{2}-1 = 1-2\sin^2\frac{k}{2} \tag{12}$$

To obtain the quantization condition we solve for $\cos k$ to get

$$\cos k = \frac{1-\tan^2\frac{k}{2}}{1+\tan^2\frac{k}{2}}=\frac{\cot\frac{\lambda}{2}+\frac{2g}{\lambda}-\tan\frac{\lambda}{2}}{\cot\frac{\lambda}{2}+\tan\frac{\lambda}{2}}=\cos\lambda+g\frac{\sin\lambda}{\lambda} \tag{13}$$

We can express the wave function as:

$$\psi(x) = e^{-ikx}u_k(x)$$

where $u_k(x)$ is a periodic function which defines in the interval $[-\frac{1}{2},\frac{1}{2}]$ will be:

$$u_k(x) = \frac{A}{2}e^{-ikx}\times\begin{cases}\cos(\lambda x)+\left(iq-\frac{g}{\lambda}\right)\sin(\lambda x) & x<0\\ \cos(\lambda x)+\left(iq+\frac{g}{\lambda}\right)\sin(\lambda x) & x>0\end{cases} \tag{14}$$

where $q = \left[\cot\frac{\lambda}{2}+\frac{g}{\lambda}\right]\tan\frac{k}{2} = \left[\tan\frac{\lambda}{2}-\frac{g}{\lambda}\right]\cot\frac{k}{2}$.

4)

Because $\cos k$ is a periodic function the whatever solution we obtain for λ will also be periodic in k therefore we will just restrict our selves in the first Brillouin zone $[-\pi,\pi]$. Because also $\cos k$ is symmetric in this interval the set of λ solutions is also going to be symmetric, so we will consider that k belongs to $[0,\pi]$.

To understand that this equation corresponds to bands we can plot the right hand side as a function of λ (ie the energy). It will start at $1+g$ at the origin and then it will fluctuate while approaching a simple cosine at large λ. Occasionally it will go "off bounds", that is outside the $[-1, 1]$ interval for some λ, in which case there is no solution. Lets consider the extrema:

$$-\sin\lambda + g\frac{\cos\lambda}{\lambda} - g\frac{\sin\lambda}{\lambda^2} = 0 \Rightarrow \tan\lambda = \frac{\lambda}{\frac{\lambda^2}{g} + 1}$$

The extreme values of the function are:

$$\cos\lambda\left(1 + \frac{g}{\lambda}\tan\lambda\right) = \cos\lambda\left(1 + \frac{g}{\frac{\lambda^2}{g} + 1}\right) = \pm\frac{g + \frac{\lambda^2}{g} + 1}{\sqrt{\lambda^2 + \left(\frac{\lambda^2}{g} + 1\right)^2}}$$

where I used: $\cos^2\lambda = \left(1 + \tan^2\lambda\right)^{-1}$. By taking the difference of the squares of the numerator and the denominator we can show that this value is always greater than one. So the right hand side of the quantization condition oscillates between extremes larger than one. Clearly in a region around the extremes there is no solution for lambda (band gaps) but once the function gets within the bounds it will continue to change monotonically until it gets out of bounds again taking therefore all possible values between -1 and 1 exactly once. Therefore each band has as many states as there are k's in the Brilluin zone ($[-\pi, \pi]$). From part two we saw that there is 1 state in every $\frac{2\pi}{N}$ so that in each Brilluin zone there are N states and this the number of states in the band. At the limit $L \to \infty$ the k states form the continuous set $[-\pi, \pi]$ and the bands are also continuous (because the right hand side is a continuous function.

We note also that for $\lambda = \pi n$ the function becomes ± 1 and this is one the the energies corresponding to the boundaries $k = \pm\pi$ of the Brilluin zone.

In the case of large g and for $\lambda \ll g$ the quantization condition reads:

$$\cos k \approx g\frac{\sin\lambda}{\lambda}$$

At the extrema $\tan\lambda = \lambda$, $\sin^2\lambda = \frac{\lambda^2}{1+\lambda^2}$ and the values of the function are $\pm\frac{g}{\sqrt{1+\lambda^2}}$ which for relatively small λ will be of the order of g. This means that the function fluctuates between large extreme values and therefore it will be very steep when it is within bounds. Another way to see this is to consider its derivative at the center of the band $\cos k = 0 \Rightarrow \sin\lambda_n = 0 \Rightarrow \lambda_n = \pi n$ for $n = 1, 2, \ldots$ which corresponds to energies

$$E_n = \frac{\pi^2 n^2}{2}$$

. The derivative is $\pm\frac{g}{\lambda}$ and is large for $g \gg \lambda$. We can approximate the right side with a Taylor series to get $\pm\frac{g}{\lambda_n}(\lambda - \lambda_n)$. The band width $\frac{g}{\lambda_n}\Delta\lambda = 2$ because $\cos k$ changes by 2 from one side of the band to the other. Using $\lambda = \sqrt{2E}$ gives $\frac{\Delta\lambda}{\lambda} = \frac{\Delta E}{2E}$ and the bandwidth is

$$\frac{\Delta E}{E} = \frac{4}{g}$$

. From 14 the wave function in the interval $[-\frac{1}{2}, \frac{1}{2}]$ becomes $\approx (i + sing(x)) \frac{\sin(\pi n x)}{\pi n}$ which is zero at the locations of the peaks x =integer.

In the opposite limit where $g \ll \lambda$ the right hand side of the quantization condition is almost $\cos\lambda$ so the divergence from the free particle spectrum will be negligible. There are going to be only few non allowed energies centered around the extrema. To get the non allowed values take the square of the quantization condition and make the right side greater than one:

$$\cos^2\lambda + \frac{2g\sin\lambda\cos\lambda}{\lambda} + \frac{g^2\sin^2\lambda}{\lambda^2} > 1$$

which after some manipulations becomes:

$$\frac{2g\cot\lambda}{\lambda} + \frac{g^2}{\lambda^2} > 1$$

If we ignore the second order terms we get $\cot\lambda > \frac{\lambda}{2g}$ and the non allowed states are the ones that are the the tails of the $\cot\lambda$ at $\lambda_n = n\pi$. In this regime the $\cot\lambda \approx \frac{1}{\lambda - \lambda_n} = \frac{1}{\Delta\lambda}$ so that

$$\Delta\lambda \approx \frac{2g}{\lambda_n}$$

which means that for large energies the width of non allowed states vanishes. When k is away from the boundaries of the Brilluin zone then we can approximate $\lambda = k$ and $g = 0$. The wave function becomes then the free particle one. This picture breaks down close the the Brilluin zone boundaries. There $\cos k$ obtains an extremum and so does λ. Then the dispersion $E = E(k)$ looks like the free particle one apart from the high symmetry points where it “breaks” so that the derivative there is zero.

4 Quantum States of the Linear Harmonic Oscillator

1)

To simplify notation we will define the dimensionless position and momentum

$$\begin{aligned} \hat{x} &= \sqrt{\frac{m\omega}{\hbar}}\hat{X} \\ \hat{p} &= \frac{1}{\sqrt{m\omega\hbar}}\hat{P} \end{aligned}$$

This is equivalent as using natural units $\hbar = m = \omega = 1$. In either case the Hamiltonian becomes:

$$H = \hbar\omega\frac{\hat{x}^2 + \hat{p}^2}{2}$$

The commutator of the dimensionless position and momentum operators is

$$[\hat{x}, \hat{p}] = i$$

The creation and annihilation operators are dimensionless and can be written as

$$\hat{a}^\dagger = \frac{\hat{x} - i\hat{p}}{\sqrt{2}} \tag{15}$$

$$\hat{a} = \frac{\hat{x} + i\hat{p}}{\sqrt{2}} \tag{16}$$

with commutator:

$$\begin{aligned}
\left[\hat{a}, \hat{a}^\dagger\right] &= \left[\frac{\hat{x} + i\hat{p}}{\sqrt{2}}, \frac{\hat{x} - i\hat{p}}{\sqrt{2}}\right] \\
&= -\frac{i}{2}\left[\hat{X}, \hat{P}\right] + \frac{i}{2}\left[\hat{P}, \hat{X}\right] \\
&= \frac{-i}{2} i + \frac{i}{2} i = 1
\end{aligned}$$

To get the second line we used the linearity of the commutator along with $[\hat{x}, \hat{x}] = 0$ and $[\hat{p}, \hat{p}] = 0$.

2)

In terms of creation annihilation operators:

$$\begin{aligned}
\hat{a}^\dagger \hat{a} &= \left(\frac{\hat{X} - i\hat{P}}{\sqrt{2}}\right)\left(\frac{\hat{X} + i\hat{P}}{\sqrt{2}}\right) \\
&= \frac{1}{2}\left(\hat{X}^2 + \hat{P}^2 - i\hat{P}\hat{X} + i\hat{X}\hat{P}\right) \\
&= \frac{1}{2}\left(\hat{X}^2 + \hat{P}^2 + i\left[\hat{X}, \hat{P}\right]\right) \\
&= H - \frac{1}{2}
\end{aligned}$$

or

$$H = \hbar\omega\left(\hat{a}^\dagger \hat{a} + \frac{1}{2}\right)$$

.

3)

If we apply the Hamiltonian on this state

$$H\left|0\right\rangle = \frac{\hbar\omega}{2}\left|0\right\rangle$$

because the term $\hat{a}^\dagger\hat{a}$ annihilates the state. So $|0\rangle$ is an eigenstate with energy $\frac{\hbar\omega}{2}$, the zero-point energy. To show that this is indeed the ground state we take the expectation value of this hamiltonian with a random wave function $|\psi\rangle$:

$$\langle \psi \left| H \right| \psi\rangle = \hbar\omega \left\langle \psi \left| \hat{a}^\dagger \hat{a} \right| \psi\right\rangle + \frac{\hbar\omega}{2}$$

The first term is proportional to the norm of $\hat{a}\,|\psi\rangle$ and is therefore ≥ 0. It is equal to zero if and only if $\hat{a}\,|\psi\rangle = 0$ and this proves that $|0\rangle$ is the ground state.

One can prove inductively that

$$\left[\hat{a}, \hat{a}^{\dagger n}\right] = n\hat{a}^{\dagger n-1}$$

It clearly holds for $n = 1$ and lets assume that it holds for n. Then we have for $n + 1$

$$\left[\hat{a}, \hat{a}^{\dagger n+1}\right] = \hat{a}^\dagger \left[\hat{a}, \hat{a}^{\dagger n}\right] + \left[\hat{a}, \hat{a}^\dagger\right] \hat{a}^{\dagger n} = n\hat{a}^\dagger \hat{a}^{\dagger n-1} + \hat{a}^{\dagger n} = (n+1)\,\hat{a}^{\dagger n}$$

This can be written as:

$$\hat{a}\hat{a}^{\dagger n} = n\hat{a}^{\dagger n-1} + \hat{a}^{\dagger n}\hat{a} \tag{17}$$

Applying the Hamiltonian to the state $|n\rangle$:

$$\begin{aligned} H\,|n\rangle &= \hbar\omega A_n \left(\hat{a}^\dagger\hat{a} + \frac{1}{2}\right) \hat{a}^{\dagger n}\,|0\rangle \\ &= \hbar\omega A_n \left(n\hat{a}^\dagger\hat{a}^{\dagger n-1} + \hat{a}^\dagger\hat{a}^{\dagger n}\hat{a} + \frac{1}{2}\hat{a}^{\dagger n}\right)|0\rangle \\ &= \hbar\omega A_n \left(n + \frac{1}{2}\right)\hat{a}^{\dagger n}\,|0\rangle = \left(n + \frac{1}{2}\right)|n\rangle \end{aligned}$$

The middle term in the second line annihilates the vacuum state. Because the Hamiltonian is Hermitian and the eigenstates correspond to different eigen energies, then they must be orthogonal:

$$\langle n|\, H\, |m\rangle = E_n \langle n|m\rangle = \langle n|m\rangle\, E_m$$

This is obtained by applying H first to the left and then to the right. For $m \neq n$ and $E_n \neq E_m$ the only possibility for the relation to hold is if $\langle n|m\rangle = 0$.

The normalization constant is evaluated using 17:

$$\begin{aligned} |A_n|^{-2} &= \langle 0|\,\hat{a}^n\hat{a}^{\dagger n}\,|0\rangle = \langle 0|\,\hat{a}^{n-1}\hat{a}\hat{a}^{\dagger n}\,|0\rangle = \\ &= \langle 0|\,\hat{a}^{n-1}\left(n\hat{a}^{\dagger n-1} + \hat{a}^{\dagger n}\hat{a}\right)|0\rangle = \\ &= n\,\langle 0|\,\hat{a}^{n-1}\hat{a}^{\dagger n-1}\,|0\rangle = n\,|A_{n-1}|^{-2} \end{aligned}$$

Because $A_0 = \langle n|n\rangle = 1$ this iteratively gives $|A_n|^{-2} = n!$.

Finally the eigenstate is:

$$|n\rangle = \frac{\hat{a}^{\dagger n}}{\sqrt{n!}} |0\rangle \tag{18}$$

An interesting property that one can derive from this expression is

$$\begin{aligned} \hat{a}\,|n\rangle &= \hat{a}\frac{\hat{a}^{\dagger n}}{\sqrt{n!}} |0\rangle = \frac{1}{\sqrt{n!}} \left(n\hat{a}^{\dagger n-1} + \hat{a}^{\dagger n}\hat{a}\right)|0\rangle = \sqrt{n}\,|n-1\rangle & (19) \\ \hat{a}^{\dagger}\,|n\rangle &= \hat{a}^{\dagger}\frac{\hat{a}^{\dagger n}}{\sqrt{n!}} |0\rangle = \sqrt{n+1}\,|n+1\rangle & (20) \end{aligned}$$

4)

Inverting equations: 15 and 16 gives

$$\begin{aligned} \hat{x} &= \frac{\hat{a}^{\dagger} + \hat{a}}{\sqrt{2}} \\ \hat{p} &= i\frac{\hat{a}^{\dagger} - \hat{a}}{\sqrt{2}} \end{aligned}$$

Also

$$\begin{aligned} x^2 &= \frac{\hat{a}^{\dagger 2} + \hat{a}^2 + \hat{a}^{\dagger}\hat{a} + \hat{a}\hat{a}^{\dagger}}{2} \\ \hat{x}^3 &= \frac{\hat{a}^{\dagger 3} + \hat{a}^3 + \hat{a}^{\dagger 2}\hat{a} + \hat{a}^2\hat{a}^{\dagger} + \hat{a}^{\dagger}\hat{a}^2 + \hat{a}\hat{a}^{\dagger 2} + \hat{a}\hat{a}^{\dagger}\hat{a} + \hat{a}^{\dagger}\hat{a}\hat{a}^{\dagger}}{2\sqrt{2}} \end{aligned}$$

It is then straightforward to show that:

1. $\left\langle n \middle| \hat{X} \middle| n' \right\rangle = \sqrt{\frac{\hbar}{m\omega}} \frac{\sqrt{n}\left\langle n-1|n'\right\rangle + \sqrt{n'}\left\langle n|n'-1\right\rangle}{\sqrt{2}} = \sqrt{\frac{\hbar}{2m\omega}}\left(\sqrt{n}\delta_{n-1,n'} + \sqrt{n'}\delta_{n,n'-1}\right)$
2. $\left\langle n \middle| \hat{P} \middle| n' \right\rangle = \sqrt{m\omega\hbar} \frac{\sqrt{n}\left\langle n-1|n'\right\rangle - \sqrt{n'}\left\langle n|n'-1\right\rangle}{i\sqrt{2}} = i\sqrt{\frac{m\omega\hbar}{2}}\left(\sqrt{n}\delta_{n-1,n'} - \sqrt{n'}\delta_{n,n'-1}\right)$
3.

$$\begin{aligned} \left\langle 3 \middle| \hat{X}^3 \middle| 2 \right\rangle &= \left(\frac{\hbar}{m\omega}\right)^{3/2} \frac{\left\langle 3 \left| \hat{a}^{\dagger 2}\hat{a} + \hat{a}\hat{a}^{\dagger 2} + \hat{a}^{\dagger}\hat{a}\hat{a}^{\dagger} \right| 2 \right\rangle}{2\sqrt{2}} \\ &= \left(\frac{\hbar}{2m\omega}\right)^{3/2} \left(\sqrt{3\cdot 2\cdot 2} + \sqrt{4\cdot 4\cdot 3} + \sqrt{3\cdot 3\cdot 3}\right) = 3\left(\frac{3\hbar}{2m\omega}\right)^{3/2} \end{aligned}$$

For the last matrix element we picked the terms with one $\hat{a}^{\dagger}$ more than $\hat{a}$ and used equations 20 and 19 repeatedly.

5)

The importance of the coherence states relies in the fact that they minimize both Δp and Δx which makes them the most classical of all quantum states.

Using a Taylor expansion and equation 18:

$$|z\rangle = \sum_{n=0}^{\infty} \frac{z^n a^{\dagger n}}{n!} |0\rangle = \sum_{n=0}^{\infty} \frac{z^n}{\sqrt{n!}} |n\rangle$$

Applying $\hat{a}$ and using equation 19 at each term in the Taylor expansion gives

$$\hat{a}\,|z\rangle = \sum_{n=0}^{\infty} \frac{z^n}{\sqrt{n!}} \hat{a}\,|n\rangle = \hat{a}\,|0\rangle + \sum_{n=1}^{\infty} \frac{z z^{n-1}}{\sqrt{n!}} \sqrt{n}\,|n-1\rangle = z\,|z\rangle \tag{21}$$

Similarly one can get $\langle z|\,\hat{a}^\dagger = \langle z|\,z^*$ with the same procedure after taking the conjugate defining equation of the coherent states.

The inner product of the two coherent states is

$$\langle z|\omega\rangle = \langle 0|\, e^{z^*\hat{a}} e^{\omega\hat{a}^\dagger}\,|0\rangle = \langle 0|\, e^{\omega\hat{a}^\dagger} e^{z^*\hat{a}} e^{z^*\omega[\hat{a},\hat{a}^\dagger]}\,|0\rangle = e^{z^*\omega}\,\langle 0|\, e^{\omega\hat{a}^\dagger} e^{z^*\hat{a}}\,|0\rangle = e^{z^*\omega}$$

where we used the Baker-Haudorf-Campbell identity for $A = \hat{a}$ and $B = \hat{a}^\dagger$.

Expressing equation 21 using spacial coordinates and the natural system of units we have

$$\left(x + \frac{\partial}{\partial x}\right)\psi(x) = \sqrt{2}z\psi(x)$$

To solve this differential equation multiply both sides with $e^{\frac{1}{2}x^2}$ and after some manipulation it will transform to:

$$\frac{\partial}{\partial x}\left[e^{\frac{1}{2}x^2}\psi(x)\right] = \sqrt{2}ze^{\frac{1}{2}x^2}\psi(x)$$

with an exponential as the general solution, thus:

$$\psi(x) \;=\; Ne^{-\frac{1}{2}x^2}e^{\sqrt{2}z}$$

For the normalization constant we evaluate

$$\frac{\langle z|z\rangle}{|N|^2} = |N|^{-2}\int_{-\infty}^{\infty} |\psi(x)|^2\,dx = \int_{-\infty}^{\infty} e^{-x^2+\sqrt{2}(z^*+z)x}dx = \sqrt{\pi}e^{\frac{1}{2}(z^*+z)^2}$$

where $\langle z|z\rangle = \exp(|z|^2)$. We can use a real normalization constant:

$$\begin{aligned} N &= \langle z|z\rangle^{\frac{1}{2}}\,\pi^{-\frac{1}{4}}e^{-\frac{1}{4}(z^*+z)^2} \\ &= \pi^{-\frac{1}{4}}e^{-\frac{1}{4}\left(z^{*2}+z^2\right)} \end{aligned}$$

Restoring the units will give the required result.

5 Bound States of the Poeschel-Teller

In this problem we will use the natural system of units

$$\hbar = m = a = 1$$

In this system $\lambda = \sqrt{2\left|E\right|}$, $2U_0 = s(s+1)$ and the Schrodinger equation becomes:

$$-\frac{\partial^2}{\partial x^2}\psi(x) + \frac{2U_0}{\cosh^2(x)}\psi(x) = 2E\psi(x)$$

1)

For $|x| \to \infty$ we can neglect the exponentially vanishing potential and for bound states get

$$-\frac{\partial^2}{\partial x^2}\psi(x) = -2\left|E\right|\psi(x) = -\lambda^2\psi(x)$$

The solutions corresponding to the bound states are the vanishing ones:

$$\psi(x) \approx Ne^{-\lambda|x|}$$

1. At $x \to +\infty$, $\tanh(x) \to 1 - 2e^{-2x}$ so that $u \to e^{-2x}$ and $\psi(x) = Ne^{-\lambda x} \to Nu^{\frac{\lambda}{2}}$. Therefore

$$\omega(u) \approx N\left(4u\left(1-u\right)\right)^{-\frac{\lambda}{2}} u^{\frac{\lambda}{2}} \approx N'$$

2. At the limit $x \to -\infty$, $\tanh(x) = -1 + 2e^{2x}$ so that $u \to 1 - e^{2x}$ and $\psi(x) = Ne^{\lambda x} \to N\left(1-u\right)^{\frac{\lambda}{2}}$. Then:

$$\omega(u) \approx N\left(4u\left(1-u\right)\right)^{-\frac{\lambda}{2}}\left(1-u\right)^{\frac{\lambda}{2}} \to N'(1-u)^{\lambda}$$

Thus in both limits $\omega(u)$ becomes a constant.

Around the origin the potential is almost constant because $\cosh(x) = 1 - \frac{x^2}{2} + \ldots$ and therefore

$$-\frac{1}{2}\frac{\partial^2}{\partial x^2}\psi(x) + U_0\psi(x) = E\psi(x)$$

The solutions here is

$$\psi(x) = N\left(e^{i\sqrt{2E-2U_0}x} \pm e^{-i\sqrt{2E-2U_0}x}\right)$$

At the origin $u \to \frac{1-x}{2}$ and thus $\left(4u\left(1-u\right)\right)^{\frac{\lambda}{2}} \to 1.$ and $\omega(u) \to N\left(e^{i\sqrt{2E-2U_0}(1-2u)} \pm e^{-i\sqrt{2E-2U_0}(1-2u)}\right.$

2)

Using the trigonometric identity

$$\cosh^{-2}(x) = 1 - \tanh^2(x) = (1 - \tanh(x))(1 + \tanh(x))$$

the potential in terms of u becomes

$$U(u) = -4s(s+1)u(1-u)$$

.

The second derivative in terms of x for some function $\phi(u)$

$$\frac{\partial}{\partial x}\frac{\partial}{\partial x}\phi(u) = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\frac{\partial \phi}{\partial u}\right) = \frac{\partial^2 u}{\partial x^2}\frac{\partial \phi}{\partial u} + \left(\frac{\partial u}{\partial x}\right)^2 \frac{\partial^2 \phi}{\partial u^2}$$

And also for some $f(u)g(u)$ with $f(u) = e^{\phi(u)}$ and $\frac{\partial u}{\partial x} = q(u)$ the second derivative is

$$\begin{aligned}
\frac{\partial^2}{\partial x^2} f(u)g(u) &= \frac{\partial^2 f}{\partial x^2} g + 2\frac{\partial f}{\partial x}\frac{\partial g}{\partial x} + f\frac{\partial^2 g}{\partial x^2} \\
&= \frac{\partial^2 f}{\partial x^2} g + \left(2\frac{\partial f}{\partial x}\frac{\partial u}{\partial x} + f\frac{\partial^2 u}{\partial x^2}\right)\frac{\partial g}{\partial u} + f\left(\frac{\partial u}{\partial x}\right)^2 \frac{\partial^2 g}{\partial u^2} \\
&= gq^2\phi' f\frac{\partial}{\partial u}\ln(qf\phi') + fq^2\left(2\phi' + \frac{\partial \ln q}{\partial u}\right)\frac{\partial g}{\partial u} + fq^2\frac{\partial^2 g}{\partial u^2} \\
&= f\left(\frac{\partial u}{\partial x}\right)^2\left[\frac{\partial^2 g}{\partial u^2} + \frac{\partial \ln(qf^2)}{\partial u}\frac{\partial g}{\partial u} + \left(\frac{\partial^2 \ln f}{\partial u^2} + \frac{\partial \ln f}{\partial u}\frac{\partial}{\partial u}\ln(qf)\right) g\right]
\end{aligned}$$

If we apply this for $f(u) = (4u(1-u))^{\frac{\lambda}{2}}$ and $g(u) = \omega(u)$ we get:

$$\begin{aligned}
q(u) = \frac{\partial u}{\partial x} &= -\frac{1}{2\cosh^2(x)} = -2u(1-u) \\
\frac{\partial \ln(qf^2)}{\partial u} &= (\lambda+1)\left(\frac{1}{u} - \frac{1}{1-u}\right) = (\lambda+1)\frac{1-2u}{u(1-u)} \\
\frac{\partial^2 \ln f}{\partial u^2} + \frac{\partial \ln f}{\partial u}\frac{\partial}{\partial u}\ln(qf) &= \frac{\lambda}{2}\left(-\frac{1}{u^2} - \frac{1}{(1-u)^2}\right) + \frac{\lambda}{2}(\frac{\lambda}{2}+1)\left(\frac{1}{u} - \frac{1}{1-u}\right)^2 \\
&= \frac{\lambda^2}{4}\frac{1}{u^2(1-u)^2} - \lambda(\lambda+1)\frac{1}{u(1-u)}
\end{aligned}$$

The Schrodinger equation

$$\frac{\partial^2}{\partial x^2}\psi(x) - \lambda^2\psi(x) - 2U(x)\psi(x) = 0$$

will become:

$$
\begin{aligned}
0 &= f\left(\frac{\partial u}{\partial x}\right)^2\left[\omega''(u)+(\lambda+1)\frac{1-2u}{u(1-u)}\omega'(u)-\lambda(\lambda+1)\frac{\omega(u)}{u(1-u)}\right] \\
&+\left[8U_0u(1-u)+\frac{\lambda^2}{4}\frac{1}{u^2\left(1-u\right)^2}\left(\frac{\partial u}{\partial x}\right)^2-\lambda^2\right]f(u)\omega(u)
\end{aligned}
$$

Because $\left(\frac{\partial u}{\partial x}\right)^2 = 4u^2(1-u^2)$ the last two terms cancel and the equation reads:

$$
u(1-u)\omega''(u)+(\lambda+1)\left(1-2u\right)\omega'(u)-\left(\lambda(\lambda+1)-s(s+1)\right)\omega(u)=0
$$

This is of the Hypergeometric form with the parameters

$$
\begin{aligned}
\alpha\beta &= \lambda(\lambda+1)-s(s+1)=(\lambda-s)(\lambda+s+1) \\
\gamma &= \lambda+1 \\
\alpha+\beta+1 &= 2(\lambda+1)
\end{aligned}
$$

It trivial to show that $\alpha=\lambda-s$ and $\beta=\lambda+s+1$.

3)

The hypergeometric equation has a general solution

$$
\omega(u)=AF(\alpha,\beta,\gamma,u)+Bu^{1-\gamma}F(\beta-\gamma+1,\alpha-\gamma+1,2-\gamma,u)
$$

The hypergeometric function $F(\alpha,\beta,\gamma,u)$ is analytic everywhere in $|u|<1$ with a Taylor series:

$$
F(\alpha,\beta,\gamma,u) = \sum_{n=0}^{\infty}c_nu^n=\sum_{n=0}^{\infty}\frac{(\alpha)_n(\beta)_n}{n!(\gamma)_n}u^n
$$

where $(\alpha)_n=\alpha(\alpha+1)(\alpha+2)\dots(\alpha+n-1)$ is the Pochhammer symbol (rising factorial).

In the limiting case $u\to 0$ both hypergeometric functions converge to a constant and $u^{1-\gamma}=u^{-\lambda}$ which diverges. Therefore in this regime the dominant term is the divergent second term. However, as we showed in the first part, $\omega(u)$ converges in to a constant in the same regime. From this we conclude that $B=0$. In the case of $u\to 1$ we showed that $\omega(u)$ converges also into a constant. The ratio of two successive terms in the hypergeometric series is:

$$
\frac{c_{n+1}}{c_n}=\frac{(\alpha+n)(\beta+n)}{(n+1)(\gamma+n)}
$$

This ratio converges to 1 for large n which is consistent to the regime of validity $|u|<1$. However if the series does not terminate then starting from some large $n = N_0 \gg \alpha, \beta, \gamma$ we can replace the coefficients of the series with $c_n \approx c_{N_0}$ so that these terms contribute a

$$\sum_{n=N_0}^{\infty} c_{N_0} u^n = c_{N_0} u^{N_0} \sum_{n=0}^{\infty} u^n = c_{N_0} u^{N_0} \frac{1}{1-u}$$

Therefore if the series does not terminate the large-n terms sum into a function that diverges for $u \to 1$. To remedy the problem we have to assume that the series terminates which means that there is some $n \geq 0$ such that:

$$(\alpha + n)(\beta + n) = 0$$

Since $\alpha = \lambda - s$ and $\beta = \lambda + s + 1$ this condition implies that either $s = \lambda + n$ or $s = -\lambda - n - 1$. Because $\lambda = 2|E| > 0$ and $n \geq 0$ this means that either $s > 0$ or $s < -1$ and therefore $2U_0 = s(s+1) > 0$. There are no bound states for $U_0 \leq 0$ and this is the mathematical proof of this physically obvious statement. Without loss of generality we will pick $s > 0$ because even if $s < -1$ we can parametrize it as $s = -1 - s'$ where $s' > 0$ and then $2U_0 = s'(s'+1)$. In this case we get

$$\lambda = s - n$$

Since $\lambda > 0$ this reads to $n < s$. There is a maximum number of allowed bound states which increases as the depth of the potential increases:

$$n_{max} = [s]$$

where $[s]$ is the floor function.

4)

The energy of the bound states is

$$E = -\frac{1}{2}(s-n)^2$$

where $n = 0, 1, 2, \ldots, [s]$ and $s > 0$ is a measure of the strength of the potential and can be expressed in terms of U_0 as

$$s = -\frac{1}{2} + \frac{1}{2}\sqrt{1 + 8U_0}$$

or restoring the original units

$$\begin{aligned} E &= -\frac{\hbar^2}{ma^2}(s-n)^2 \\ s &= -\frac{1}{2} + \frac{1}{2}\sqrt{1 + \frac{8mU_0}{\hbar^2 a^2}} \end{aligned}$$