1 I. Wave Packets

1.

$$\begin{aligned}
\langle\Phi|\Phi\rangle &= \int_{\infty}^{\infty} dp\langle\Phi|p\rangle\langle p|\Phi\rangle \\
&= A^2\int_{\infty}^{\infty} dpe^{-\frac{(p-p_0)^2}{2\sigma}} = A^2\sqrt{2\sigma\pi} = 1 \\
\therefore \quad & A = \frac{1}{\sqrt[4]{2\sigma\pi}}
\end{aligned}$$

2.

$$\begin{aligned}
\langle x|\Phi\rangle &= \int_{\infty}^{\infty} dp\langle x|p\rangle\langle p|\Phi\rangle \\
&= A\int_{\infty}^{\infty} \frac{dp}{\sqrt{2\pi\hbar}}e^{ipx/\hbar}e^{-\frac{(p-p_0)^2}{4\sigma}} \\
&= \frac{1}{\sqrt[4]{2\sigma\pi}}\frac{1}{\sqrt{2\pi\hbar}}\sqrt{4\pi\sigma}e^{-\frac{\sigma(x+a)^2}{\hbar^2}}e^{i\frac{p_0(x+a)}{\hbar}} \\
&= \sqrt[4]{\frac{2\sigma}{\pi\hbar^2}}e^{-\frac{\sigma(x+a)^2}{\hbar^2}}e^{i\frac{p_0(x+a)}{\hbar}}
\end{aligned}$$

3.

$$\begin{aligned}
\langle\Phi|\hat{P}|\Phi\rangle &= \int_{\infty}^{\infty} dp\int_{\infty}^{\infty} dp'\langle\Phi|p\rangle\langle p|\hat{P}|p'\rangle\langle p'|\Phi\rangle = \int_{\infty}^{\infty} dp\int_{\infty}^{\infty} dp'\langle\Phi|p\rangle p\delta(p-p')\langle p'|\Phi\rangle \\
&= \int_{\infty}^{\infty} dp\langle\Phi|p\rangle p\langle p|\Phi\rangle = \int_{\infty}^{\infty} dp\Phi_p^* p\Phi_p \\
&= A^2\int_{\infty}^{\infty} dppe^{-\frac{(p-p_0)^2}{2\sigma}} = p_0 \\
Similarly, \langle\Phi|\hat{X}|\Phi\rangle &= \int_{\infty}^{\infty} dx\Phi_x^* x\Phi_x \\
&= \sqrt{\frac{2\sigma}{\pi\hbar^2}}\int_{\infty}^{\infty} dx\, xe^{-\frac{2\sigma(x+a)^2}{\hbar^2}} = -a
\end{aligned}$$

4.

$$\begin{aligned}
(\Delta X)^2 &= \langle(x-\overline{x})^2\rangle = \langle x^2\rangle - \overline{x}^2; (\Delta P)^2 = \langle P^2\rangle - \overline{p}^2 \\
\langle\Phi|\hat{X}^2|\Phi\rangle &= \int_{\infty}^{\infty} dx\Phi_x^* x^2\Phi_x \\
&= \sqrt{\frac{2\sigma}{\pi\hbar^2}}\int_{\infty}^{\infty} dxx^2e^{-\frac{2\sigma(x+a)^2}{\hbar^2}}
\end{aligned}$$

$$\begin{aligned}
&= \sqrt{\frac{2\sigma}{\pi\hbar^2}}\int_{\infty}^{\infty} dy(y^2-2ay+a^2)e^{-\frac{2\sigma y^2}{\hbar^2}} \\
&= \sqrt{\frac{2\sigma}{\pi\hbar^2}}\int_{\infty}^{\infty} dyy^2e^{-\frac{2\sigma y^2}{\hbar^2}}+a^2 \\
&= \sqrt{\frac{2\sigma}{\pi\hbar^2}}(\frac{\hbar^2}{2\sigma})^{3/2}\Gamma(\frac{3}{2})+a^2 \\
\therefore (\Delta X)^2 &= \sqrt{\frac{2\sigma}{\pi\hbar^2}}(\frac{\hbar^2}{2\sigma})^{3/2}\Gamma(\frac{3}{2}) = \frac{\hbar^2}{4\sigma} \\
\langle\Phi|\hat{P}^2|\Phi\rangle &= A^2\int_{\infty}^{\infty} dp\, p^2 e^{-\frac{(p-p_0)^2}{2\sigma}} = \frac{2\sigma}{\sqrt{\pi}}\Gamma(3/2)+p_0^2 \\
\therefore (\Delta P)^2 &= \sigma;\, \Delta X\Delta P = \sqrt{\frac{\hbar^2}{4\sigma}\sigma} = \frac{\hbar}{2} \geq \frac{\hbar}{2}
\end{aligned}$$

The equality is satisfied only when $(1)\Delta\hat{P}|\Phi\rangle \propto \Delta\hat{X}|\Phi\rangle, (2)\langle\Phi|[\Delta\hat{X},\Delta\hat{P}]_+|\Phi\rangle = 0$, following is the details:
(1)

$$\begin{aligned}
\Delta\hat{P}|\Phi\rangle &= (\hat{p}-p_0)|\Phi\rangle = (-i\hbar\frac{d}{dx}-p_0)[Ae^{-\sigma(x+a)^2/\hbar^2}e^{ip_0(x+a)/\hbar}] \\
&= [-i\hbar(\frac{-2\sigma(x+a)}{\hbar^2}+\frac{ip_0}{\hbar})-p_0]|\Phi\rangle \\
&= \frac{2i\sigma(x+a)}{\hbar}|\Phi\rangle \\
Meanwhile,\ \Delta\hat{X}|\Phi\rangle &= (\hat{x}-\langle x\rangle)|\Phi\rangle = (x+a)|\Phi\rangle \propto \Delta\hat{P}|\Phi\rangle
\end{aligned}$$

(2)

$$\begin{aligned}
\langle\Phi|[\Delta\hat{X},\Delta\hat{P}]_+|\Phi\rangle &= \langle\Phi|[\hat{p}-p_0,\hat{x}+a]_+|\Phi\rangle \\
&= \langle\Phi|[\hat{p},\hat{x}]_+ - 2p_0\hat{x}+2a\hat{p}-2p_0a|\Phi\rangle \\
&= \langle\Phi|[\hat{p},\hat{x}]+2\hat{x}\hat{p}|\Phi\rangle + 2p_0a+2p_0a-2p_0a \\
&= -i\hbar+2\langle\Phi|\hat{x}\hat{p}|\Phi\rangle+2p_0a \\
&= -i\hbar+2\langle\Phi|x(\frac{2i\sigma(x+a)}{\hbar}+p_0)|\Phi\rangle+2p_0a \\
&= -i\hbar+\frac{4i\sigma}{\hbar}(\frac{\hbar^2}{4\sigma}+a^2-a^2)-2p_0a+2p_0a \\
&= 0
\end{aligned}$$

2 II States in a one-Dimensional Oscillator

1. It is easy to prove $\overline{X}=0;\overline{P}=0$, thus $(\Delta X)^2 = \langle\Phi|\hat{X}^2|\Phi\rangle; (\Delta P)^2 = \langle\Phi|\hat{P}^2|\Phi\rangle$

$$\langle\Phi|\hat{H}|\Phi\rangle = \frac{1}{2m}\langle\Phi|\hat{P}^2|\Phi\rangle + \frac{m\omega^2}{2}\langle\Phi|\hat{X}^2|\Phi\rangle$$

$$= \frac{(\Delta P)^2}{2m} + \frac{m\omega^2}{2}(\Delta X)^2 \geq \omega(\Delta P \Delta X) \geq \frac{\hbar\omega}{2}$$

The Inequality becomes equality only when:
(1)$(\hat{P} - \overline{P})|\Phi\rangle = c(\hat{X} - \overline{X})|\Phi\rangle$, (2)$\langle\Phi|[\hat{P}, \hat{X}]_+|\Phi\rangle = 0$, (3)$\frac{(\Delta P)^2}{2m} = \frac{m\omega^2}{2}(\Delta X)^2$,
let us check each one:
(1) $\hat{P}|\Phi\rangle = -i\hbar\frac{\partial}{\partial x} = \frac{i\hbar}{2a^2}x|\Phi\rangle = c\hat{X}|\Phi\rangle$;
(2) $\langle\Phi|[\hat{P}, \hat{X}]_+|\Phi\rangle = \langle\Phi|[\hat{P}, \hat{X}] + 2\hat{X}\hat{P}|\Phi\rangle = -i\hbar + 2\langle\Phi|\hat{X}\hat{P}|\Phi\rangle = -i\hbar + \frac{i\hbar}{a^2}a^2 = 0$
(3)$\Delta P = \frac{\hbar}{2\Delta X} \Rightarrow \frac{\hbar^2}{8m(\Delta X)^2} = \frac{m\omega^2}{2}(\Delta X)^2 \Rightarrow \Delta X = \sqrt{\frac{\hbar}{2m\omega}}$
Meanwhile $(\Delta X)^2 = \langle\Phi|X^2|\Phi\rangle = a^2$;Thus $a = \sqrt{\frac{\hbar}{2m\omega}}$

2.

$$\begin{aligned}
\langle\Phi|[\hat{T}, \hat{V}]|\Phi\rangle &= \frac{\omega^2}{4}\langle\Phi|\hat{P}^2, \hat{X}^2|\Phi\rangle \\
&= \frac{\omega^2}{4}\langle\Phi| - 2i\hbar(\hat{X}\hat{P} + \hat{P}\hat{X})|\Phi\rangle \\
&= \frac{-i\hbar\omega^2}{2}\langle\Phi|[\hat{X}, \hat{P}]_+|\Phi\rangle
\end{aligned}$$

3. Set $\Delta\hat{T} = \hat{T} - \langle T\rangle; \Delta\hat{V} = \hat{V} - \langle V\rangle$, according to Shankar's, we can get

$$\begin{aligned}
(\Delta T)^2(\Delta V)^2 &\geq \frac{1}{4}|\langle\Phi|[\Delta\hat{T}, \Delta\hat{V}]_+|\Phi\rangle|^2 + \frac{1}{4}|\langle\Phi|[\hat{T}, \hat{V}]|\Phi\rangle|^2 \\
&= \frac{1}{4}|\langle\Phi|[\Delta\hat{T}, \Delta\hat{V}]_+|\Phi\rangle|^2 + \frac{\hbar^2\omega^4}{16}|\langle\Phi|[\hat{X}, \hat{P}]_+|\Phi\rangle|^2
\end{aligned}$$

Based on Part 1(2), $\langle\Phi|[\hat{X}, \hat{P}]_+|\Phi\rangle = 0$, meanwhile, $\langle T\rangle = \frac{\hbar^2}{8ma^2}$ and $\langle V\rangle = \frac{m\omega^2 a^2}{2}$

$$\begin{aligned}
\langle\Phi|[\Delta\hat{T}, \Delta\hat{V}]_+|\Phi\rangle &= \langle TV + VT\rangle - 2\langle T\rangle\langle V\rangle \\
&= \frac{-\hbar^2\omega^2}{4}\left[2\langle\Phi|\Phi\rangle + 4\langle\Phi|X\frac{d}{dx}|\Phi\rangle + 2\langle\Phi|X^2\frac{d^2}{dX^2}|\Phi\rangle\right] - \frac{\hbar^2\omega^2}{8} \\
&= \frac{-\hbar^2\omega^2}{4}[2 - 4 \times \frac{1}{2} + 2 \times \frac{1}{4}] - \frac{\hbar^2\omega^2}{8} = -\frac{\hbar^2\omega^2}{4}
\end{aligned}$$

Thus $\Delta T\Delta V \geq \frac{\hbar^2\omega^2}{8}$ and the equality is true when $\Delta\hat{T}|\Phi\rangle \propto \Delta\hat{V}|\Phi\rangle$, we can check here:

$$\begin{aligned}
\Delta\hat{T}|\Phi\rangle &= (\hat{T} - \langle T\rangle)|\Phi\rangle \\
&= [\frac{-\hbar^2}{2m}(\frac{-1}{2a^2} + \frac{X^2}{4a^4}) - \frac{\hbar^2}{8ma^2}]|\Phi\rangle \\
&= \frac{\hbar^2}{8ma^4}(a^2 - X^2)|\Phi\rangle
\end{aligned}$$

$$while,\ \Delta\hat{V}|\Phi\rangle = \frac{m\omega^2}{2}(x^2-a^2)|\Phi\rangle;$$

Thus, $\Delta\hat{T}|\Phi\rangle \propto \Delta\hat{V}|\Phi\rangle$ and it is true that $(\Delta T)(\Delta V) = \frac{\hbar^2\omega^2}{8}$

3 III Equation of Motion of Operators

1. Set the time evolution operator $U(t) = e^{\int -i\hat{H}dt/\hbar}$;
$\hat{X}(t) = U^+(t)XU(t);\ \ \hat{P}(t) = U^+(t)PU(t).$
$|\Phi\rangle_S = U(t)|\Phi\rangle_{t=0} = U(t)|\Phi\rangle_H$

2.

$$\begin{aligned}
i\hbar\frac{d\hat{X}(t)}{dt} &= [\hat{X}(t),\hat{H}] = [\hat{X}(t),\frac{\hat{P}^2}{2m}] = i\hbar\frac{\hat{P}(t)}{m} \\
\therefore \frac{d\hat{X}(t)}{dt} &= \frac{\hat{P}(t)}{m} \\
i\hbar\frac{d\hat{P}(t)}{dt} &= [\hat{P}(t),\hat{H}] = [\hat{P}(t),\frac{1}{2}m\omega^2\hat{X}^2 + f(x)\hat{X}] \qquad (1)\\
&= -i\hbar m\omega^2\hat{X} + f(t)(-i\hbar) \\
\therefore \frac{d\hat{P}}{dt} &= -m\omega^2\hat{X} - f(t); \\
\therefore \frac{d^2\hat{X}(t)}{dt^2} &+ \omega^2\hat{X} = -\frac{f(t)}{m}
\end{aligned}$$

Which is just the equation of motion for the harmonic oscillator with external force.

4 IV Quantum Measurement with Photons

1. Set $|x\rangle = (1,0); |y\rangle = (0,1); |R\rangle = \frac{1}{\sqrt{2}}(1,+i); |L\rangle = \frac{1}{\sqrt{2}}(1,-i); |\psi\rangle = (a,b)$
Thus $\hat{P}_x = \frac{\hat{I}+\hat{\sigma}_3}{2} = \begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix}, \hat{P}_y = \frac{\hat{I}-\hat{\sigma}_3}{2} = \begin{pmatrix}0 & 0\\ 0 & 1\end{pmatrix}, \hat{P}_R = \frac{\hat{I}+\hat{\sigma}_2}{2} = \frac{1}{2}\begin{pmatrix}1 & -i\\ i & 1\end{pmatrix}, \hat{P}_L = \frac{\hat{I}-\hat{\sigma}_2}{2} = \frac{1}{2}\begin{pmatrix}1 & i\\ -i & 1\end{pmatrix}$

$$\begin{aligned}
\langle\psi|\hat{P}_x|\psi\rangle &= (a^*,b^*)\begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix}\binom{a}{b} = a^*a \\
P_x &= |\langle x|\psi\rangle|^2 = a^*a; \\
\langle\psi|\hat{P}_y|\psi\rangle &= (a^*,b^*)\begin{pmatrix}0 & 0\\ 0 & 1\end{pmatrix}\binom{a}{b} = b^*b \\
P_y &= |\langle y|\psi\rangle|^2 = b^*b;
\end{aligned}$$

Similarly, we could prove that $P_R = |\langle R|\psi\rangle|^2 = \langle\psi|\hat{P}_R|\psi\rangle$, $P_L = |\langle L|\psi\rangle|^2 = \langle\psi|\hat{P}_L|\psi\rangle$, Actually, based on the definition of projection operator.

$$\begin{aligned}
\hat{P}_x &= |x\rangle\langle x| = (\begin{smallmatrix}1\\0\end{smallmatrix})(1,0) = \begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix}\\
\hat{P}_y &= |y\rangle\langle y| = (\begin{smallmatrix}0\\1\end{smallmatrix})(0,1) = \begin{pmatrix}0 & 0\\ 0 & 1\end{pmatrix}\\
\hat{P}_R &= |R\rangle\langle R| = \frac{1}{2}(\begin{smallmatrix}1\\i\end{smallmatrix})(1,-i) = \frac{1}{2}\begin{pmatrix}1 & -i\\ +i & 1\end{pmatrix}\\
\hat{P}_L &= |L\rangle\langle L| = \frac{1}{2}(\begin{smallmatrix}1\\-i\end{smallmatrix})(1,+i) = \frac{1}{2}\begin{pmatrix}1 & +i\\ -i & 1\end{pmatrix}
\end{aligned}$$

2.

$$\begin{aligned}
P_x &= Tr(\hat{P}_x\hat{\rho}) = Tr\left[\begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix}\begin{pmatrix}a & c^*\\ c & b\end{pmatrix}\right] = a\\
P_y &= Tr(\hat{P}_y\hat{\rho}) = Tr\left[\begin{pmatrix}0 & 0\\ 0 & 1\end{pmatrix}\begin{pmatrix}a & c^*\\ c & b\end{pmatrix}\right] = b\\
P_R &= Tr(\hat{P}_R\hat{\rho}) = \frac{1}{2}Tr\left[\begin{pmatrix}1 & -i\\ i & 1\end{pmatrix}\begin{pmatrix}a & c^*\\ c & b\end{pmatrix}\right] = \frac{1}{2}[a+b+i(c^*-c)] = \frac{1}{2}[1+i(c^*-c)]\\
P_L &= Tr(\hat{P}_L\hat{\rho}) = \frac{1}{2}Tr\left[\begin{pmatrix}1 & i\\ -i & 1\end{pmatrix}\begin{pmatrix}a & c^*\\ c & b\end{pmatrix}\right] = \frac{1}{2}[a+b+i(c-c^*)] = \frac{1}{2}[1+i(c-c^*)]
\end{aligned}$$

3. $\langle\hat{L}_z\rangle = Tr(\hat{L}_z\hat{\rho}) = \hbar Tr\left[\begin{pmatrix}0 & -i\\ i & 0\end{pmatrix}\begin{pmatrix}a & c^*\\ c & b\end{pmatrix}\right] = i\hbar(c^*-c)$;

4. $Tr(\hat{\rho}^2) = Tr\left[\begin{pmatrix}cos^2\theta & e^{i\Phi}sin\theta cos\theta\\ e^{-i\Phi}sin\theta cos\theta & sin^2\theta\end{pmatrix}^2\right] = cos^4\theta + 2cos^2\theta sin^2\theta + sin^4\theta = (cos^2\theta + sin^2\theta)^2 = 1$

Thus this density matrix represents a pure state and we can set $|\psi\rangle = a|x\rangle + b|y\rangle = (a,b)$;

$\hat{\rho} = |\psi\rangle\langle\psi| = (\begin{smallmatrix}a\\b\end{smallmatrix})(a^*,b^*) = \begin{pmatrix}aa^* & ab^*\\ a^*b & b^*b\end{pmatrix} = \begin{pmatrix}cos^2\theta & e^{i\Phi}sin\theta cos\theta\\ e^{-i\Phi}sin\theta cos\theta & sin^2\theta\end{pmatrix}$

Therefore,$a = cos\theta$, $b = sin\theta e^{-i\Phi}$ and $|\psi\rangle = cos\theta|x\rangle + sin\theta e^{-i\Phi}|y\rangle$

$L_z = i\hbar(c^*-c) = i\hbar(e^{-i\Phi} - e^{i\Phi})sin\theta cos\theta = 2i\hbar sin\theta cos\theta sin\Phi = \hbar sin2\theta sin\Phi$