Physics 580: Quantum Mechanics I
Department of Physics, UIUC
Fall Semester 2006
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Problem Set No. 2:
Review of Classical Mechanics
Due Date: September 18, 2006

1 The Least Action Principle

Consider a mechanical system with a degree of freedom $\vec{q}$, and let $L(\vec{q}, \dot{\vec{q}})$ be the Lagrangian for this system . Use the Least Action Principle to derive the Euler-Lagrange equation for this system, *i. e.* the *equation of motion* for $\vec{q}(t)$. Note: we did this is class; you are supposed to go over the details of the derivation.

2 Poisson Brackets

Consider first a generic dynamical system with coordinates $\{q_i\}$ and momenta $\{p_i\}$, where $i = 1, \ldots, N$. The Poisson Bracket for a pair of dynamical observables $A[q, p]$ and $B[q, p]$ is defined as

$$\{A, B\}_{qp} \equiv \sum_{i=1}^{N} \left(\frac{\partial A}{\partial q_i} \frac{\partial B}{\partial p_i} - \frac{\partial A}{\partial p_i} \frac{\partial B}{\partial q_i} \right)$$

1. Use Hamilton's equations to show that the following equation of motion is correct
$$\frac{dA}{dt} = \frac{\partial A}{\partial t} + \{A, H\}$$

2. Derive the fundamental Poisson Brackets
$$\{q_i, p_j\} = \delta_{ij} \qquad \{q_i, q_j\} = 0 \qquad \{p_i, p_j\} = 0$$

3. Compute the Poisson Brackets
$$\{q_i, A\} \quad \text{and} \quad \{p_i, A\}$$
where $A \equiv A[q, p]$ is a generic physical observable.

4. Consider a single particle in three dimensional space. Let us denote $(x, y, z) = (x_1, x_2, x_3)$. Compute the Poisson Brackets
$$\{x_i, L_j\} \quad \text{and} \quad \{p_i, L_j\}$$

where $L_j = (\vec{x} \wedge \vec{p})_j = \epsilon_{jkl} x_k p_l$ is the angular momentum. Here ϵ_{jkl} is the Levi-Civita tensor

$$\epsilon_{jkl} = \begin{cases} 1 & \text{if} \quad (jkl) = (1,2,3) \quad \text{and cyclic permutations} \\ -1 & \text{if} \quad (jkl) = (2,1,3) \quad \text{and cyclic permutations} \\ 0 & \qquad\qquad\qquad\qquad\qquad \text{otherwise} \end{cases}$$

3 Charged Particle in an Electromagnetic Field

In class we discussed the Lagrangian description of the problem of the motion of a charged particle of charge q and mass m, in an external electromagnetic field given in terms of the scalar potential $\phi(\vec{x}, t)$, and vector potential $\vec{A}(\vec{x}, t)$, where $\vec{x}(t)$ is the position vector of the charged particle at time t. The Lagrangian for this system is

$$L(\vec{x}, \dot{\vec{x}}) = \frac{1}{2} m \dot{\vec{x}}^2 - q\phi(\vec{x}, t) + \frac{q}{c}\, \dot{\vec{x}} \cdot \vec{A}(\vec{x}, t)$$

where c is the speed of light in vacuum.

1. Find the momentum $\vec{p}$ conjugate to the coordinate $\vec{x}$.

2. Find the equation of motion for $\vec{x}$. Write this equation only in terms of the electric field $\vec{E} = -\vec{\bigtriangledown}\phi - \frac{1}{c}\frac{\partial}{\partial t}\vec{A}$, and of the magnetic field $\vec{B} = \vec{\bigtriangledown} \times \vec{A}$.

3. Show that the *Hamiltonian* $H(\vec{x}, \vec{p})$ for this system is given by

$$H(\vec{x}, \vec{p}) = \frac{1}{2m}\left(\vec{p} - \frac{q}{c}\vec{A}\right)^2 + q\phi$$

4. You will consider now the special case in which the particle is restricted to move on the xy plane and there is a uniform magnetic field of strength B perpendicular to the plane. In the "circular" gauge, $\vec{\bigtriangledown} \cdot \vec{A} = 0$, the components of the vector potential are given by

$$A_x = -\frac{B}{2} y \qquad A_y = \frac{B}{2} x \qquad A_z = 0$$

   From now on we will set $z = 0$. We will also assume that the in-plane electric field is zero, and set $\phi = 0$. Let us define the generalized coordinates Q_1, Q_2 and momenta P_1 and P_2 by

$$Q_1 = \frac{cp_x}{qB} + \frac{y}{2} \qquad P_1 = p_y - \frac{qB}{2c} x$$

$$Q_2 = \frac{cp_y}{qB} + \frac{x}{2} \qquad P_2 = p_x - \frac{qB}{2c} y$$

   Show that these new coordinates and momenta are canonical. Write the Hamiltonian in terms of Q's and P's.

5. Consider again the same situation as in the last item. Show that it has the form of a linear harmonic oscillator. Determine the angular frequency ω of this oscillator, and give a physical interpretation for this frequency. One of the coordinates Q is cyclic. What does this tell you about its conjugate momentum? Does the this Hamiltonian depend on that momentum? Why? Why not? Give a physical interpretation of these results.

4 Symmetries and Conservation Laws

Consider the two-body problem in three space dimensions. Let $\vec{r}_1$ and $\vec{r}_2$ be the position vectors of two particles of masses m_1 and m_2 respectively. The Lagrangian of this system is

$$L = \frac{1}{2} m_1 \dot{\vec{r}}_1^{\,2} + \frac{1}{2} m_2 \dot{\vec{r}}_2^{\,2} - V(|\vec{r}_1 - \vec{r}_2|)$$

where $V(r_{12})$ is a pair interaction that depends only on the distance $r_{12} = |\vec{r}_1 - \vec{r}_2|$ between the two particles.

1. Show that in the coordinates

$$\vec{r} = \vec{r}_1 - \vec{r}_2 \qquad \vec{R} = \frac{m_1 \vec{r}_1 + m_2 \vec{r}_2}{m_1 + m_2}$$

   the Lagrangian takes the form

$$L = \frac{1}{2} M \dot{\vec{R}}^{\,2} + \frac{1}{2} \mu \dot{\vec{r}}^{\,2} - V(|\vec{r}|)$$

   where $M = m_1 + m_2$ is the total mass, and $\mu = m_1 m_2/(m_1 + m_2)$ is the reduced mass.

2. Write this Lagrangian for this system in spherical coordinates for the relative coordinate $\vec{r}$. Find the Hamiltonian in these coordinates.

3. What coordinates are cyclic in these coordinates? What are the conservation laws expressed by these facts? Justify your statements.

4. Using Poisson Brackets show that $\vec{L}$, the angular momentum with respect to the center of mass, is the generator of infinitesimal rotations. Use Poisson Brackets to show that for a central potential $\vec{L}$ is conserved.

5. Is the radial component of the linear relative momentum conserved? Justify your answers using Poisson Brackets.

6. Consider now the case $V(r) = -\kappa/r$. Show that, in addition of the angular momentum $\vec{L}$, the Runge-Lenz vector $\vec{M}$

$$\vec{M} = \frac{1}{\mu} \vec{p} \wedge \vec{L} - \frac{\kappa}{r} \vec{r}$$

   is also a constant of motion. Use Poisson Brackets to show that this is true.