Problem 1

Follow the lecture notes.

Problem 2

1)

The Hamilton's equations are

$$\begin{aligned} \dot{p}_i &= -\frac{\partial H}{\partial q_i} \\ \dot{q} &= \frac{\partial H}{\partial p_i} \end{aligned}$$

For the general observable $A = A(p_i, q_i, t)$ and the time derivative is:

$$\begin{aligned} \frac{dA}{dt} &= \sum_i \left(\frac{\partial A}{\partial q_i}\dot{q}_i + \frac{\partial A}{\partial p_i}\dot{p}_i \right) + \frac{\partial A}{\partial t} \\ &= \sum_i \left(\frac{\partial A}{\partial q_i}\frac{\partial H}{\partial p_i} - \frac{\partial A}{\partial p_i}\frac{\partial H}{\partial p_i} \right) + \frac{\partial A}{\partial t} \\ &= \{A, H\} + \frac{\partial A}{\partial t} \end{aligned}$$

2)

Remember that p_i and q_i are treated as completely independent variables.

$$\begin{aligned} \{q_i, p_j\} &= \sum_k \left(\frac{\partial q_i}{\partial q_k}\frac{\partial p_j}{\partial p_k} - \frac{\partial q_i}{\partial p_k}\frac{\partial p_j}{\partial q_k} \right) = \sum_k \delta_{ik}\delta_{jk} - 0 \cdot 0 = \delta_{ij} \\ \{q_i, p_j\} &= \sum_k \left(\frac{\partial q_i}{\partial q_k}\frac{\partial q_j}{\partial p_k} - \frac{\partial q_i}{\partial p_k}\frac{\partial q_j}{\partial q_k} \right) = \sum_k \delta_{ik} \cdot 0 - 0 \cdot \delta_{jk} = 0 \\ \{p_i, p_j\} &= \sum_k \left(\frac{\partial p_i}{\partial q_k}\frac{\partial p_j}{\partial p_k} - \frac{\partial p_i}{\partial p_k}\frac{\partial p_j}{\partial q_k} \right) = \sum_k 0 \cdot \delta_{jk} - \delta_{ik} \cdot 0 = 0 \end{aligned}$$

3)

$$\begin{aligned} \{q_i, A\} &= \sum_k \left(\frac{\partial q_i}{\partial q_k}\frac{\partial A}{\partial p_k} - \frac{\partial q_i}{\partial p_k}\frac{\partial A}{\partial q_k} \right) = \sum_k \left(\delta_{ik}\frac{\partial A}{\partial p_k} - 0 \cdot \frac{\partial A}{\partial q_k} \right) = \frac{\partial A}{\partial p_i} \\ \{p_i, A\} &= \sum_k \left(\frac{\partial p_i}{\partial q_k}\frac{\partial A}{\partial p_k} - \frac{\partial p_i}{\partial p_k}\frac{\partial A}{\partial q_k} \right) = \sum_k \left(0 \cdot \frac{\partial A}{\partial p_k} - \delta_{ik}\frac{\partial A}{\partial q_k} \right) = -\frac{\partial A}{\partial q_i} \end{aligned}$$

4)

Using the results of the previous question for $A \to L_j$ and $q \to x_i$ we get

$$\begin{aligned}
\{x_i, L_j\} &= \frac{\partial L_j}{\partial p_i} = \varepsilon_{jkl} x_k \frac{\partial p_l}{\partial p_i} = \varepsilon_{jkl} x_k \delta_{li} = \varepsilon_{jki} x_k = \varepsilon_{ijk} x_k \\
\{p_i, L_j\} &= -\frac{\partial L_j}{\partial x_i} = \varepsilon_{jkl} \left(-\frac{\partial x_k}{\partial x_i} \right) p_l = -\varepsilon_{jkl} \delta_{ki} p_l = -\varepsilon_{jil} p_l = \varepsilon_{ijl} p_l
\end{aligned}$$

Problem 3

In this problem we will use the shorthand notation:

$$\begin{aligned}
\nabla_{\vec{x}} &= \hat{x}\frac{\partial}{\partial x} + \hat{y}\frac{\partial}{\partial y} + \hat{z}\frac{\partial}{\partial z} \\
\nabla_{\dot{\vec{x}}} &= \hat{x}\frac{\partial}{\partial \dot{x}} + \hat{y}\frac{\partial}{\partial \dot{y}} + \hat{z}\frac{\partial}{\partial \dot{z}}
\end{aligned}$$

with $\hat{x}$, $\hat{y}$ and $\hat{z}$ being the unit vectors.

1)

$$\vec{p} = \nabla_{\dot{\vec{x}}} L(\vec{x}, \dot{\vec{x}}) = m\dot{\vec{x}} + \frac{q}{c}\vec{A}(\vec{x}, t) \tag{1}$$

2)

Use the Euler-Lagrange equations for the problem:

$$\frac{d}{dt}\nabla_{\dot{\vec{x}}} L = \nabla_{\vec{x}} L$$

The right side is

$$\nabla_{\vec{x}} L = -q\nabla_{\vec{x}}\phi + \frac{q}{c}\nabla_{\vec{x}}\left(\dot{\vec{x}} \cdot \vec{A}(\vec{x}, t)\right)$$

Using Eq. **??**, the left side reads:

$$\frac{d}{dt}\left(m\dot{\vec{x}} + \frac{q}{c}\vec{A}(\vec{x}, t)\right) = m\ddot{\vec{x}} + \frac{q}{c}\left[\left(\dot{\vec{x}} \cdot \nabla_{\vec{x}}\right)\vec{A}(\vec{x}, t) + \frac{\partial}{\partial t}\vec{A}(\vec{x}, t)\right]$$

Substituting in the Euler-Lagrange equations and bringing everything but the acceleration term to the right gives:

$$m\ddot{\vec{x}} = \left(-q\nabla_{\vec{x}}\phi - \frac{q}{c}\frac{\partial}{\partial t}\vec{A}(\vec{x}, t)\right) + \left[\frac{q}{c}\left(\dot{\vec{x}} \cdot \nabla_{\vec{x}}\right)\vec{A}(\vec{x}, t) - \frac{q}{c}\nabla_{\vec{x}}\left(\dot{\vec{x}} \cdot \vec{A}(\vec{x}, t)\right)\right]$$

The first term of the right side is $q\vec{E}$. It can be shown that the second term is $\dot{\vec{x}} \times \vec{B} = \dot{\vec{x}} \times \left(\nabla \times \vec{A}(\vec{x},t)\right)$ by working backwards (repeated indices imply summation):

$$\begin{aligned} \dot{\vec{x}} \times \left(\nabla \times \vec{A}(\vec{x},t)\right) &= \vec{e}_k \varepsilon_{ijk} \dot{x}_i \left(\varepsilon_{abj} \partial_a A_b\right) \\ &= \left(\varepsilon_{ijk}\varepsilon_{abj}\right) \vec{e}_k \dot{x}_i \partial_a A_b \end{aligned}$$

In the sum $\varepsilon_{ijk}\varepsilon_{abj}$, either $i = a$ and $k = b$ in which case $\varepsilon_{ijk}\varepsilon_{ikj} = -1$ or $i = b$ and $k = a$ in which case $\varepsilon_{ijk}\varepsilon_{kij} = 1$. This is summarized as follows:

$$\varepsilon_{ijk}\varepsilon_{abj} = \delta_{ib}\delta_{ka} - \delta_{ia}\delta_{kb}$$

Thus:

$$\begin{aligned} \dot{\vec{x}} \times \left(\nabla \times \vec{A}(\vec{x},t)\right) &= \vec{e}_k \dot{x}_i \partial_k A_i - \vec{e}_k \dot{x}_i \partial_i A_k \\ &= \nabla_{\vec{x}} \left(\dot{\vec{x}} \cdot \vec{A}\right) - \left(\dot{\vec{x}} \cdot \nabla_{\vec{x}}\right) \vec{A} \end{aligned}$$

Therefore the equation of motion takes the familiar form:

$$m\ddot{\vec{x}} = q\vec{E} + \frac{q}{c}\dot{\vec{x}} \times \vec{B}$$

3)

$$\begin{aligned} H &= \dot{\vec{x}} \cdot \vec{p} - \frac{1}{2}m\dot{\vec{x}}^2 + q\phi(\vec{x}) - \frac{q}{c}\dot{\vec{x}} \cdot \vec{A}(\vec{x},t) \\ &= \dot{\vec{x}} \cdot \left(\vec{p} - \frac{q}{c}\vec{A}(\vec{x},t)\right) - \frac{1}{2}m\dot{\vec{x}}^2 + q\phi(\vec{x}) \\ &= \frac{1}{2}m\dot{\vec{x}}^2 + q\phi(\vec{x}) \\ &= \frac{1}{2m}\left(\vec{p} - \frac{q}{c}\vec{A}(\vec{x},t)\right)^2 + q\phi(\vec{x}) \end{aligned}$$

To go from the second to the third line we solved Eq. **??** for $\dot{\vec{x}}$.

4)

One obvious and tedious way is to use the fact that the original coordinates are canonical and thus $\{p_i, p_j\} = 0$, $\{x_i, x_j\} = 0$, $\{x, p_y\} = \{y, p_x\} = 0$ and $\{x, p_x\} = \{y, p_x\} = -\{p_x, x\} = -\{p_y, x\} = 1$ and also that the Poisson brackets are linear in both arguments and go and calculate all possible Poisson brackets of the new coordinates. This can be done in symplectic (matrix) formalism too but in any case it is the same procedure.

Another way is to show that there is a generating function $F(x, y, Q_1, Q_2, t)$ which gives the transformation. The constraints are:

$$\begin{aligned}
p_x &= \frac{\partial F}{\partial x} = \frac{qB}{c}\left(Q_1 - \frac{y}{2}\right) \\
p_y &= \frac{\partial F}{\partial y} = \frac{qB}{c}\left(Q_2 - \frac{x}{2}\right) \\
-P_1 &= \frac{\partial F}{\partial Q_1} = \frac{qB}{c}\left(-Q_1 + x\right) \\
-P_2 &= \frac{\partial F}{\partial Q_2} = \frac{qB}{c}\left(-Q_2 + y\right)
\end{aligned}$$

It is easy to demonstrate that this transformation can be generated by the function:

$$\frac{c}{qB}F(x, y, Q_1, Q_2) = -\frac{Q_1^2 + Q_2^2 + xy}{2} + xQ_1 + yQ_2$$

Therefore the new coordinates must be canonical.

5)

$$\begin{aligned}
H &= \frac{1}{2m}\left(p_x + \frac{qB}{2c}y\right)^2 + \frac{1}{2m}\left(p_y - \frac{qB}{2c}x\right)^2 \\
&= \frac{1}{2m}\left(\frac{qB}{c}\right)^2 Q_1^2 + \frac{1}{2m}P_1^2
\end{aligned}$$

This hamiltonian has the form of a simple harmonic oscillator involving only the coordinates Q_1 and P_1. Comparing with the Hamiltonian of the simple 1-dimensional harmonic oscillator

$$H = \frac{p^2}{2m} + \frac{m\omega^2}{2}x^2$$

we obtain a frequency

$$\omega = \frac{qB}{mc}$$

which is the frequency that corresponds to the circular motion of a charged particle in a magnetic field.

The Q_2 coordinate does not appear in the hamiltonian which makes it cyclic. However P_2 also does not appear in the hamiltonian. This does not mean that one degree of freedom was eliminated from the system because the Q_2 and P_2 are necessary for the canonical transformation to be invertible. However from the canonical equations of motion one gets $\dot{P}_2 = 0$ and $\dot{Q}_2 = 0$, that is both Q_2 and P_2 are integrals of motion. To demonstrate what this means let $Q_2 = x_0$ and $P_2 = -\frac{qB}{c}y_0$. Then

$$(p_x, p_y) = \frac{qB}{2c}(-y + 2y_0, x - 2x_0) \tag{2}$$

Next lets solve for Q_1 and P_1. This is the standard harmonic oscillator solution:

$$\begin{aligned} Q_1 &= A_0\cos(\omega t+\phi_0) \\ P_1 &= -\frac{qB}{c}A_0\sin(\omega t+\phi_0) \end{aligned}$$

Where A_0 is the amplitude and is a measure of the energy.

Substituting in the equations of the transformation that give Q_1 and P_1 and solving for p_i gives:

$$(p_x,p_y)=\frac{qB}{2c}(y-2A_0\sin(\omega t+\phi_0),-x+2A_0\cos(\omega t+\phi_0)) \tag{3}$$

By eliminating the p_i from Eq. **??** and **??** we get the solution for the equations of motion:

$$(x,y)=(x_0+A_0\cos(\omega t+\phi_0),y_0+A_0\sin(\omega t+\phi_0))$$

Therefore the Q_2 and P_2 are indeed related to the coordinates of the center of the circular motion of the charged particle and they must be constants of motion. Notice that the magnetic field essentially localises a charged particle.

Problem 4

1)

Inverting the transformation will give

$$\begin{aligned} \vec{r_1} &= \vec{R}+\frac{m_2}{m_1+m_2}\vec{r} \\ \vec{r_2} &= \vec{R}-\frac{m_1}{m_1+m_2}\vec{r} \end{aligned}$$

The kinetic energy part of the Lagrangian will transform as follows:

$$\begin{aligned} \frac{1}{2}m_1\dot{\vec{r_1}}^2+\frac{1}{2}m_2\dot{\vec{r_2}}^2 &= \frac{1}{2}m_1\left(\dot{\vec{R}}+\frac{m_2}{m_1+m_2}\dot{\vec{r}}\right)^2+\frac{1}{2}m_2\left(\dot{\vec{R}}-\frac{m_1}{m_1+m_2}\dot{\vec{r}}\right)^2 \\ &= \frac{1}{2}\left(m_1+m_2\right)\dot{\vec{R}}^2+\frac{1}{2}\left[m_1\left(\frac{m_2}{m_1+m_2}\right)^2+m_2\left(\frac{m_1}{m_1+m_2}\right)^2\right]\dot{\vec{r}}^2 \\ &= \frac{1}{2}\left(m_1+m_2\right)\dot{\vec{R}}^2+\frac{1}{2}\frac{m_1m_2}{m_1+m_2}\dot{\vec{r}}^2 \end{aligned}$$

which is the required result for the kinetic energy.

2)

The Lagrangian can be broken in two independent parts $L=L_{CM}+L_{rel}$ where

$$
\begin{aligned}
L_{CM} &= \frac{1}{2} M \dot{\vec{R}}^2 \\
L_{rel} &= \frac{1}{2}\mu\left(\dot{r}^2 + r^2\dot{\theta}^2 + r^2\cos^2(\theta)\dot{\phi}^2\right) - V(r)
\end{aligned}
$$

The Lagrangian is of the form $L = T - V$ which means that the Hamiltonian will simply be $T + V$. The conjugate momenta are:

$$
\begin{aligned}
\vec{P} &= \frac{\partial L_{CM}}{\partial \dot{\vec{R}}} = M\dot{\vec{R}} \\
p_r &= \frac{\partial L_{rel}}{\partial \dot{r}} = \mu\dot{r} \\
p_\theta &= \frac{\partial L_{rel}}{\partial \dot{\theta}} = \mu r^2\dot{\theta} \\
p_\phi &= \frac{\partial L_{rel}}{\partial \dot{\phi}} = \mu r^2\sin^2(\theta)\dot{\phi}
\end{aligned}
$$

Expressing the time derivatives in terms of the moment and replacing in the kinetic energy will give the Hamiltonian $H = H_{CM} + H_{rel}$

$$
\begin{aligned}
H_{CM} &= \frac{1}{2} M \dot{\vec{R}}^2 \\
H_{rel} &= \frac{1}{2\mu}\left(p_r^2 + \frac{p_\theta^2}{r^2} + \frac{p_\phi^2}{r^2\sin^2(\theta)}\right) + V(r)
\end{aligned}
$$

3)

Since the interaction between the particles only depends on their relative distance, translating both particles at the same distance and direction does not affect the physics and therefore $\vec{R}$ must be a cyclic coordinate. The corresponding conserved momentum is the total momentum of the system. Also rotations around the z axis of the coordinate system (for any z) should not change the energy and so ϕ is a cyclic coordinate. The corresponding conservation law is the angular momentum conservation.

4)

Lets consider a simple illustrative example. Lets prove that the momentum is the generator of infinitesimal translations. Consider a function of only the position $f(q)$ (the following algebra will remain the same even if f depends on momentum). The Poisson Bracket with momentum is just $\{f(q), p\} = \frac{\partial f}{\partial q}$. Now consider the values of the function at a coordinate system translated forward by an infinitesimal amount da or $f(q+da) \approx f(q) + da\frac{\partial f}{\partial q}$. The difference in the two coordinate systems is just $da\frac{\partial f}{\partial q} = da\left\{f(q), p\right\}$ which means that the momentum can generate infinitesimal translations. In this exercise it is requested to provide a similar argument for rotations.

Consider an arbitrary function $W(\vec{r}, \vec{p})$ (the following algebra will be the same even if it depends on time) and an infinitesimal rotation:

$$\begin{aligned} \vec{r'} &= \vec{r} + \vec{da} \times \vec{r} \\ \vec{p'} &= \vec{p} + \vec{da} \times \vec{p} \end{aligned}$$

where $d\vec{a}$ is a vector with direction the axis of rotation and with magnitude the infinitesimal angle of rotation. Notice that the rotation changes all vectors (momenta and position) in the same way. Now we want to compare the values of the function in the transformed coordinate system with the values in the original system. Lets take their difference:

$$\begin{aligned} W(\vec{r'}, \vec{p'}) - W(\vec{r}, \vec{p}) &= \left(\vec{da} \times \vec{r}\right) \cdot \nabla_{\vec{r}} W(\vec{r}, \vec{p}) + \left(\vec{da} \times \vec{p}\right) \cdot \nabla_{\vec{p}} W(\vec{r}, \vec{p}) \\ &= \vec{da} \cdot \left(\vec{r} \times \nabla_{\vec{r}} + \vec{p} \times \nabla_{\vec{p}}\right) W(\vec{r}, \vec{p}) \end{aligned}$$

Now consider the Poisson Bracket of this function with the angular momentum $L_k = \varepsilon_{abk} x_a p_b$:

$$\begin{aligned} \{W(\vec{r}, \vec{p}), L_k\} &= \sum_{i=1}^{3} \frac{\partial W}{\partial x_i}\frac{\partial L_k}{\partial p_i} - \frac{\partial W}{\partial p_i}\frac{\partial L_k}{\partial x_i} \\ &= \sum_{i=1}^{3} \frac{\partial W}{\partial x_i}\varepsilon_{aik} x_a - \frac{\partial W}{\partial p_i}\varepsilon_{ibk} p_b \\ &= \sum_{i=1}^{3} \varepsilon_{aik} x_a \frac{\partial W}{\partial x_i} + \varepsilon_{bik} p_b \frac{\partial W}{\partial p_i} \\ &= \left(\vec{r} \times \nabla_{\vec{r}}\right)_k W + \left(\vec{p} \times \nabla_{\vec{p}}\right)_k W \end{aligned}$$

and therefore

$$\left\{W(\vec{r}, \vec{p}), \vec{L}\right\} = \left(\vec{r} \times \nabla_{\vec{r}} + \vec{p} \times \nabla_{\vec{p}}\right) W(\vec{r}, \vec{p})$$

This means that the angular momentum is the generator of (infinitesimal) rotations.

The Hamiltonian H_{rel} is rotationally invariant because $\vec{r} \times \nabla_{\vec{r}} V(r) = \vec{r} \times \frac{\vec{r}}{r}\frac{\partial V}{\partial r} = 0$ and $\vec{p} \times \nabla_{\vec{p}} \frac{p^2}{2} = \vec{p} \times \vec{p} = 0$. Therefore its Poisson Bracket with the angular momentum is zero and the angular momentum is conserved.

5)

It suffices to evaluate the following Poisson Bracket

$$\begin{aligned}
\{p_r, H_{rel}\} &= \left\{p_r, \frac{1}{2\mu}\left(p_r^2 + \frac{p_\theta^2}{r^2} + \frac{p_\phi^2}{r^2\cos^2(\theta)}\right) + V(r)\right\} \\
&= \frac{1}{2\mu}\left\{p_r, \frac{1}{r^2}\right\}\left(p_\theta^2 + \frac{p_\phi^2}{\cos^2(\theta)}\right) + \{p_r, V(r)\} \\
&= \frac{1}{2\mu}\frac{2}{r^3}\left(p_\theta^2 + \frac{p_\phi^2}{\cos^2(\theta)}\right) - \frac{\partial V}{\partial r}
\end{aligned}$$

Clearly it does not vanish which means that p_r cannot be conserved. If it conserved then the distance between the two particles would increase at a constant rate which could only happen if $p_\theta = p_\phi = 0$ and there is no force between them, as it would happen in a non interacting one dimensional system. However this is not the general case.

6)

We denote $H_{rel} = \frac{p^2}{2\mu} - \frac{\kappa}{r}$ the part of the Hamiltonian involving the coordinates relative to the CM. Using some standard properties of the Poisson Brackets:

$$\begin{aligned}
\{H_{CM}, M_k\} &= \frac{1}{\mu}\varepsilon_{ilk}\{H_{CM}, p_i L_j\} - \left\{H_{CM}, \frac{\kappa x_k}{r}\right\} \\
&= \frac{1}{\mu}\varepsilon_{ilk}\{H_{CM}, p_i\} L_j - \{H_{CM}, x_k\}\frac{\kappa}{r} - \left\{H_{CM}, \frac{\kappa}{r}\right\} x_k \\
&= -\frac{\kappa}{\mu}\varepsilon_{ilk}\left\{\frac{1}{r}, p_i\right\} L_j - \frac{\kappa}{\mu}\left\{\frac{p^2}{2}, x_k\right\}\frac{1}{r} - \frac{\kappa}{\mu}\left\{\frac{p^2}{2}, \frac{1}{r}\right\} x_k
\end{aligned}$$

In the second line we used the fact that $\{A, BC\} = \{A, B\} C + \{A, C\} B$ and also that the angular momentum is conserved. Also we have

$$\begin{aligned}
\left\{p_i, \frac{1}{r}\right\} &= -\frac{\partial}{\partial x_i}\frac{1}{r} = \frac{x_i}{r^3} \\
\left\{x_k, \frac{p^2}{2}\right\} &= \frac{\partial}{\partial p_k}\frac{p^2}{2} = p_k \\
\left\{\frac{p^2}{2}, \frac{1}{r}\right\} &= p_j\left\{p_j, \frac{1}{r}\right\} = \frac{\vec{p}\cdot\vec{r}}{r^3}
\end{aligned}$$

Substituting above

$$\{H_{CM}, M_k\} = \frac{\kappa}{\mu}\frac{\left(\vec{r}\times\vec{L}\right)_k}{r^3} + \frac{\kappa}{\mu}p_k\frac{1}{r} - \frac{\kappa}{\mu}\frac{\vec{p}\cdot\vec{r}}{r^3}x_k$$

Using $\vec{r} \times \vec{L} = \vec{r} \times (\vec{r} \times \vec{p}) = (\vec{r} \cdot \vec{p})\, \vec{r} - r^2 \vec{p}$ the above result reads to zero and the Runge-Lenz vector is a constant of motion.

The importance of this symmetry is that for inverse square distance forces such as the gravitational and electrostatic forces one has three integrals of motions: the energy, and angular momentum and the Runge-Lenz vector. Therefore the equations of motion are first order differential equations which can be integrated easily. In QM one can obtain the spectrum of the Hydrogen atom from symmetries only without ever invoking the Schroediger equation.