

**Physics 580: Quantum Mechanics I**  
**Department of Physics, UIUC**  
**Fall Semester 2006**  
**Professor Eduardo Fradkin**

**Problem Set No. 1:**  
**Mathematical Background**  
**Due Date: September 5, 2006**

1. Consider the set of all 3-tuples of the form  $(a, b, c)$  whose entries are real numbers. Addition and scalar multiplication are defined as

$$(a, b, c) + (d, e, f) = (a + d, b + e, c + f)$$

and

$$\alpha(a, b, c) = (\alpha a, \alpha b, \alpha c)$$

Write down the null vector and the inverse of  $(a, b, c)$ . Show that vectors of the form  $(a, b, 1)$  do not form a vector space.

2. Consider three elements from the vector space of  $2 \times 2$  real matrices

$$|1\rangle = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \quad |2\rangle = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \quad |3\rangle = \begin{pmatrix} -2 & -1 \\ 0 & -2 \end{pmatrix}$$

Are they linearly independent? Give details of your statements.

3. Let  $\mathbb{V}$  be the vector space of  $2 \times 2$  complex matrices. Let  $|A\rangle$  and  $|B\rangle$  be two  $2 \times 2$  complex matrices,  $A \in \mathbb{V}$  and  $B \in \mathbb{V}$ . Let us define  $\langle A|B\rangle$  by

$$\langle A|B\rangle \equiv \text{tr} \{A^\dagger B\}$$

Here  $\text{tr} A$  denotes the trace of the matrix  $A$ , and  $A^\dagger$  is the adjoint of  $A$ .

- (a) Show that  $\langle A|B\rangle$  defines an inner product in  $\mathbb{V}$ , i. e. check that it satisfies the axioms.
- (b) Let  $E_{ij}$  be the  $2 \times 2$  matrices with matrix elements  $(E_{ij})_{kl} = \delta_{ik}\delta_{jl}$ , where  $i, j, k, l = 1, 2$ , respectively. Show that the set of matrices  $\{E_{ij}\}$  is an orthonormal basis with respect to the inner product defined above.
- (c) Let  $A = \sum_{i,j=1,2} a_{ij} E_{ij}$ . Compute the norm of  $A$ .
4. (a) Show that the set of  $2 \times 2$  Hermitian matrices form a vector space. What conditions do their matrix elements satisfy? What is the dimension of this vector space?

(b) Consider now a general  $2 \times 2$  Hermitian matrix of the form

$$|n\rangle = \begin{pmatrix} n_0 + n_3 & n_1 - in_2 \\ n_1 + in_2 & n_0 - n_3 \end{pmatrix}$$

Compute the trace and the determinant of this matrix. Compute also its eigenvalues and its eigenvectors (with unit norm).

(c) Using that the inner product of two (Hermitian) matrices  $A$  and  $B$  is  $\langle A|B\rangle = \text{tr}(A^\dagger B)$ , form an orthonormal basis in of the vector space of  $2 \times 2$  Hermitian matrices starting with the following matrices

$$|1\rangle = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, |2\rangle = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}, |3\rangle = \begin{pmatrix} 0 & 1-i \\ 1+i & 0 \end{pmatrix}, |4\rangle = \begin{pmatrix} 1 & -i \\ i & 1 \end{pmatrix}$$

5. Let  $H$  be a Hermitian operator. Show that

$$e^{iH} \equiv \sum_{n=0}^{\infty} \frac{i^n}{n!} H^n$$

is a unitary operator.

6. Show that the trace and the determinant of a linear Hermitian operator  $\Omega$ , represented by a finite rank  $N \times N$  matrix, are not affected by a unitary change of basis  $U$ .

7. By considering the commutator, show that the Hermitian matrices  $\Omega$  and  $\Lambda$ ,

$$\Omega = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 1 \end{pmatrix}, \Lambda = \begin{pmatrix} 2 & 1 & 1 \\ 1 & 0 & -1 \\ 1 & -1 & 2 \end{pmatrix}$$

can be simultaneously diagonalized. Find the eigenvectors common to both. Verify that under a unitary transformation to this basis both matrices are diagonalized.

8. Let  $\mathbb{F}$  be the set of square integrable complex functions of a single variable  $x$ ,  $-\infty \leq x \leq +\infty$ , where by square integrable we mean that

$$\int_{-\infty}^{+\infty} dx |f(x)|^2 < \infty$$

Let  $|f\rangle \in \mathbb{F}$  and  $|g\rangle \in \mathbb{F}$ . Let us define

$$\langle f|g\rangle \equiv \int_{-\infty}^{+\infty} dx \overline{f(x)} g(x)$$

where  $\overline{f(x)}$  denotes the complex conjugate of  $f(x)$ .

(a) Show that  $\mathbb{F}$  is a vector space.

- (b) Show that  $\langle f|g \rangle$  satisfies the axioms for an inner product.  
 (c) Consider the linear differential operator

$$D = -i \frac{d}{dx}$$

acting on the vectors in  $\mathbb{F}$ . Find what are the possible boundary conditions at  $x \rightarrow \pm\infty$  that the functions in  $\mathbb{F}$  should satisfy for the operator  $D$  to be Hermitian.

- (d) Show that the vectors  $|p\rangle$

$$|p\rangle = \frac{1}{\sqrt{2\pi}} e^{ipx}$$

where  $p$  an arbitrary real number, form an orthonormal basis with respect to the inner product defined above. Use the notation

$$\delta(p - p') = \frac{1}{2\pi} \int_{-\infty}^{+\infty} dx e^{i(p - p')x}$$

for the Dirac delta-function. Show that this basis is also complete.

- (e) Check the validity of the following properties of the Dirac delta-function.
- i.

$$\delta(ax) = \frac{1}{|a|} \delta(x)$$

where  $a \in \mathbb{R}$  and  $a \neq 0$ .

- ii.

$$\delta(f(x)) = \sum_n \frac{1}{\left| \frac{df}{dx_n} \right|} \delta(x - x_n)$$

where  $x_n$  are the zeros of  $f(x)$ .