

1 I

$|a\rangle + |0\rangle = |a\rangle \implies |0\rangle = (0, 0, 0)$;
 $|a\rangle + |-a\rangle = 0; \implies |-a\rangle = (-a, -b, -c)$
 Since $(a_1, b_1, 1) + (a_1, b_1, 1) = (a_1 + a_2, b_1 + b_2, 2) \notin (a, b, 1)$,
 the vectors of the form $(a, b, 1)$ do not form a vector space.

2 II

Set $a|1\rangle + b|2\rangle + |3\rangle = 0$, then

$$\begin{cases} b - 2c = 0 \\ a + b - c = 0 \end{cases} \implies a : b : c = -1 : 2 : 1$$
 Thus we can show that $-|1\rangle + 2|2\rangle + |3\rangle = 0$ ie. they are linearly dependent.

3 III

(a) Let us check that whether 1) $\langle A|B\rangle = \langle B|A\rangle^*$, 2) $\langle A|A\rangle \geq 0$, 3) $\langle A|\mu B + \nu C\rangle = \mu\langle A|B\rangle + \nu\langle A|C\rangle$

$$\begin{aligned} \langle A|B\rangle &= \text{tr}\{A^+B\} = (A^+)_{ij}B_{ji} = A_{ji}^*B_{ji} = A_{ij}^*B_{ij} \\ \langle B|A\rangle &= \text{tr}\{B^+A\} = (B^+)_{ij}A_{ji} = B_{ji}^*A_{ji} = (A_{ij}^*B_{ij})^* \\ \langle A|B\rangle &= \langle B|A\rangle^* \end{aligned} \tag{1}$$

$$2) \langle A|A\rangle = (A^+)_{ij}A_{ji} = A_{ji}^*A_{ji} \geq 0$$

$$3) \langle A|\mu B + \nu C\rangle = \text{tr}\left(A^+(\mu B + \nu C)\right) = \mu \text{tr}(A^+B) + \nu \text{tr}(A^+C) = \mu\langle A|B\rangle + \nu\langle A|C\rangle$$

(b) $\langle E_{ij}|E_{mn}\rangle = (E_{ij})_{kl}^*(E_{mn})_{kl} = \delta_{ik}\delta_{jl}\delta_{mk}\delta_{nl} = \delta_{im}\delta_{jn}\delta_{ik}\delta_{jl} = \delta_{im}\delta_{jn}$
 Thus $\{E_{ij}\}$ is an orthonormal basis with respect to the inner product defined above.

$$(c) |A| = \sqrt{\langle A|A\rangle} = \sqrt{a_{ij}^*a_{mn}\langle E_{ij}|E_{mn}\rangle} = \sqrt{a_{ij}^*a_{mn}\delta_{im}\delta_{jn}} = \sqrt{a_{ij}^*a_{ij}}$$

4 IV

(a) For the Hermitian matrices, $A^+ = A$, ie $A_{ij} = A_{ji}^*$; set $\mu, \nu \in \mathbb{R}$
 thus $(\mu A + \nu B)_{ij}^+ = (\mu A + \nu B)_{ji}^* = \mu A_{ji}^* + \nu B_{ji}^* = \mu A_{ij} + \nu B_{ij} = (\mu A + \nu B)_{ij} \implies (\mu A + \nu B)^+ = (\mu A + \nu B)$, ie,
 It is easy to prove other axioms and thus the Hermitian matrices form a vector space. Obviously, the dimension for this vector space is 4.

$$\text{Since } \begin{pmatrix} a & b - ic \\ b + ic & d \end{pmatrix} = a \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + b \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + c \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} + d \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

(b)

$$\begin{aligned} \text{Tr}|n\rangle &= (n_0 + n_3) + (n_0 - n_3) = 2n_0 \\ \text{Det} \begin{pmatrix} n_0 + n_3 & n_1 - in_2 \\ n_1 + in_2 & n_0 - n_3 \end{pmatrix} &= n_0^2 - n_1^2 - n_2^2 - n_3^2 \end{aligned} \tag{2}$$

As for the eigenvalues and eigenvectors, $\begin{vmatrix} n_0 + n_3 - \lambda & n_1 - in_2 \\ n_1 + in_2 & n_0 - n_3 - \lambda \end{vmatrix} = 0$

Thus $\lambda = n_0 \pm \sqrt{n_1^2 + n_2^2 + n_3^2}$,

When $\lambda_1 = n_0 + \sqrt{n_1^2 + n_2^2 + n_3^2}$, $\phi'_1 = (n_1 - in_2, \sqrt{n_1^2 + n_2^2 + n_3^2} - n_3)$

Set $n_1 = n \sin \theta \cos \psi$, $n_2 = n \sin \theta \sin \psi$, $n_3 = n \cos \theta$, thus $\phi'_1 = (n \sin \theta e^{i\psi}, n(1 - \cos \theta))$

After normalization, $\phi_1 = \left(\cos \frac{\theta}{2} e^{i\psi}, \sin \frac{\theta}{2} \right)$

Similarly, for $\lambda_2 = n_0 - \sqrt{n_1^2 + n_2^2 + n_3^2}$, $\phi'_2 = ((1 - \cos \theta) e^{-i\psi}, \sin \theta)$.

After normalization, $\phi_2 = \left(\sin \frac{\theta}{2} e^{-i\psi}, \cos \frac{\theta}{2} \right)$

(c)

$$|1'\rangle = \frac{|1\rangle}{\sqrt{\langle 1|1\rangle}} = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}; \quad (3)$$

$$\begin{aligned} |2\rangle_1 &= |2\rangle - |1'\rangle \langle 1'|2\rangle = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} - \frac{1}{2} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \text{Tr} \left[\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \right] \\ &= \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \end{aligned}$$

$$|2'\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix};$$

$$\begin{aligned} |3\rangle_1 &= |3\rangle - |1'\rangle \langle 1'|3\rangle - |2'\rangle \langle 2'|3\rangle \\ &= \begin{pmatrix} 0 & 1-i \\ 1+i & 0 \end{pmatrix} - \frac{1}{2} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \text{Tr} \left[\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1-i \\ 1+i & 0 \end{pmatrix} \right] - \frac{1}{2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \text{Tr} \left[\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 0 & 1-i \\ 1+i & 0 \end{pmatrix} \right] \\ &= \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \end{aligned}$$

$$|3'\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

$$|4'\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

5 V

Generally, we can prove that $e^{\hat{A}} e^{\hat{B}} = e^{\hat{A}+\hat{B}}$, only if when $[\hat{A}, \hat{B}] = 0$;

$$\begin{aligned} e^{\hat{A}} e^{\hat{B}} &= \left(\sum_{n=0}^{\infty} \frac{A^n}{n!} \right) \left(\sum_{m=0}^{\infty} \frac{B^m}{m!} \right) = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{A^n}{n!} \frac{B^m}{m!} \\ &= \sum_{k=0}^{\infty} \frac{1}{k!} \sum_{n=0}^k \frac{k!}{n!(k-n)!} A^n B^{k-n} = \sum_{k=0}^{\infty} \frac{1}{k!} \sum_{n=0}^k C_k^n A^n B^{k-n} \end{aligned} \quad (4)$$

$$= \sum_{k=0}^{\infty} \frac{1}{k!} (A+B)^k = e^{A+B}$$

However, $(A+B)^k = \sum_{n=0}^k C_k^n A^n B^{k-n}$ is true only when $[\hat{A}, \hat{B}] = 0$. Thus, it is not difficult to show that: $[e^{iH}]^+ e^{iH} = e^{-iH} e^{iH} = e^{iH-iH} = I$, ie. it is a unitary operator.

6 VI

$$\begin{aligned} Tr\Omega' &= Tr(U^+\Omega U) = Tr(UU^+\Omega) = Tr(\Omega) \\ Det(\Omega') &= Det(U^+\Omega U) = DetU DetU^+ Det(\Omega) = Det(\Omega) \end{aligned} \tag{5}$$

7 VII

$$\begin{aligned} [\Omega, \Lambda] &= \begin{pmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} 2 & 1 & 1 \\ 1 & 0 & -1 \\ 1 & -1 & 2 \end{pmatrix} - \begin{pmatrix} 2 & 1 & 1 \\ 1 & 0 & -1 \\ 1 & -1 & 2 \end{pmatrix} \begin{pmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} 3 & 0 & 3 \\ 0 & 0 & 0 \\ 3 & 0 & 3 \end{pmatrix} - \begin{pmatrix} 3 & 0 & 3 \\ 0 & 0 & 0 \\ 3 & 0 & 3 \end{pmatrix} \\ &= 0 \end{aligned} \tag{6}$$

Thus the hermitian matrices Ω and Λ can be simultaneously diagonalized. Now let us find the eigenvalues and eigenvectors for Ω and Λ :

$$\begin{array}{l} \left| \begin{array}{ccc} 1-\omega & 0 & 1 \\ 0 & -\omega & 0 \\ 1 & 0 & 1-\omega \end{array} \right| = 0 \\ \omega_{1,2} = 0; \omega_3 = 2 \\ \omega_1 = 0, \phi_1 = \frac{1}{\sqrt{2}}(1, 0, -1); \\ \omega_2 = 0, \phi_1 = (0, 1, 0); \\ \omega_3 = 2, \phi_1 = \frac{1}{\sqrt{2}}(1, 0, 1); \end{array} \quad \begin{array}{l} \left| \begin{array}{ccc} 1-\lambda & 0 & 1 \\ 0 & \lambda & 0 \\ 1 & 0 & 1-\lambda \end{array} \right| = 0 \\ \lambda_1 = -1, \lambda_2 = 2, \lambda_3 = 3 \\ \lambda_1 = -1, \psi = \frac{1}{\sqrt{6}}(1, -2, -1); \\ \lambda_2 = 2, \psi = \frac{1}{\sqrt{3}}(1, 1, -1); \\ \lambda_3 = 3, \psi = \frac{1}{\sqrt{2}}(1, 0, -1); \end{array}$$

Since for Ω , there are two degenerate eigenvectors, we choose the eigenvectors of Λ as the eigenvectors common

to both (check!). Thus the unitary transformation matrix is $U = \begin{pmatrix} \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} \\ \frac{-2}{\sqrt{6}} & \frac{1}{\sqrt{3}} & 0 \\ \frac{-1}{\sqrt{6}} & \frac{-1}{\sqrt{3}} & \frac{1}{\sqrt{2}} \end{pmatrix}$

$$\Omega' = U^+\Omega U = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 2 \end{pmatrix} \quad \Lambda' = U^+\Lambda U = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}$$

8 VIII

a) Now let us prove $|f\rangle + |g\rangle \in \mathbb{F}$ ie is square integrable.

$$\begin{aligned}
 \int_{-\infty}^{\infty} dx |f(x) + g(x)|^2 &= \int_{-\infty}^{\infty} dx \left(|f(x)|^2 + |g(x)|^2 + f(x)^*g(x) + f(x)g(x)^* \right) \\
 &= \int_{-\infty}^{\infty} dx \left(|f(x)|^2 + |g(x)|^2 + 2\text{Re}[f(x)^*g(x)] \right) \\
 &\leq \int_{-\infty}^{\infty} dx \left(|f(x)|^2 + |g(x)|^2 + 2|f(x)||g(x)| \right) \\
 &\leq \int_{-\infty}^{\infty} dx \left(|f(x)|^2 + |g(x)|^2 + |f(x)|^2 + |g(x)|^2 \right) \\
 &= \int_{-\infty}^{\infty} dx 2 \left(|f(x)|^2 + |g(x)|^2 \right) < \infty
 \end{aligned} \tag{7}$$

Thus $|f\rangle + |g\rangle \in \mathbb{F}$ and it is easy to prove other axioms to show \mathbb{F} is a vector space.

$$b) \langle f|g\rangle = \int_{-\infty}^{\infty} dx \overline{f(x)}g(x) = \left(\int_{-\infty}^{\infty} dx \overline{g(x)}f(x) \right)^* = \langle g|f\rangle^*$$

$$\langle f|f\rangle = \int_{-\infty}^{\infty} dx \overline{f(x)}f(x) = \int_{-\infty}^{\infty} dx |f(x)|^2 \geq 0$$

$$\langle f|\mu g + \nu k\rangle = \int_{-\infty}^{\infty} dx \overline{f(x)} \left(\mu g(x) + \nu k(x) \right) = \mu \langle f|g\rangle + \nu \langle f|k\rangle$$

c) In order to make operator D to be Hermitian, $\langle f|Dg\rangle = \langle D^+ f|g\rangle$, where $D = D^+$

$$\begin{aligned}
 \langle f|Dg\rangle &= \int_{-\infty}^{\infty} dx \overline{f(x)} \left(-i \frac{d}{dx} \right) g(x) = -i \int_{-\infty}^{\infty} dx \frac{d}{dx} \left[\overline{f(x)}g(x) \right] + \int_{-\infty}^{\infty} dx \left[-i \frac{d}{dx} \overline{f(x)} \right] g(x) \\
 &= -i \overline{f(x)}g(x) \Big|_{-\infty}^{\infty} + \int_{-\infty}^{\infty} dx \left[-i \frac{d}{dx} \overline{f(x)} \right] g(x)
 \end{aligned} \tag{8}$$

If the boundary condition satisfy $f(\infty) = f(-\infty) = g(\infty) = g(-\infty) = 0$, $D^+ = -i \frac{d}{dx} = D$ ie. it is Hermitian operator.

d) $\langle p'|p\rangle = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{i(p-p')x} dx = \delta(p-p')$, the vectors $|p\rangle$ form an orthonormal basis.

For an arbitrary state $|f\rangle$, since

$$\begin{aligned}
 |f\rangle &= \int_{-\infty}^{\infty} dx f(x) |x\rangle = \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} \frac{dp}{\sqrt{2\pi}} f(p) e^{ipx} |x\rangle \\
 &= \int_{-\infty}^{\infty} dp f(p) \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} dx e^{ipx} |x\rangle = \int_{-\infty}^{\infty} dp f(p) |p\rangle
 \end{aligned} \tag{9}$$

Thus this basis is also complete.

e) When $a > 0$, $\int_{-\infty}^{\infty} dx \delta(ax) = \int_{-\infty}^{\infty} d\frac{y}{a} \delta(y) = \frac{1}{a}$
 $\therefore \delta(ax) = \frac{1}{a} \delta(x)$
 similarly, When $a < 0$, $\delta(ax) = -\frac{1}{a} \delta(x)$
 Thus, $\delta(ax) = \frac{1}{|a|} \delta(x)$

f) If $f(x) \neq 0$, $\delta(f(x)) = 0$, It is infinity only when $f(x) = 0$; Set $y = f(x)$, then $dx = \frac{1}{\frac{df}{dx}} dy$

$$\int_{-\infty}^{\infty} dx \delta(f(x)) = \sum_n \int_{x_n - \epsilon}^{x_n + \epsilon} dx \delta(f(x)) = \sum_n \int_{-\epsilon'_n}^{\epsilon'_n} \frac{1}{|\frac{df}{dx_n}|} dy \delta(y) = \sum_n \frac{1}{|\frac{df}{dx_n}|}$$

$$\therefore \delta(f(x)) = \sum_n \frac{1}{|\frac{df}{dx_n}|} \delta(x - x_n)$$