Physics 561, Fall Semester 2009
Professor Eduardo Fradkin

Problem Set No. 1:
Quantization of Non-Relativistic Fermi Systems
Due Date: September 20, 2009

1 Second Quantization of an Elastic Solid

Consider a three-dimensional elastic solid in the continuum harmonic approximation. The classical Lagrangian for this system is

$$L = \int d^3x \left\{ \sum_{i=1}^{3} \frac{\rho}{2} \dot{u}_i^2(\vec{x},t) - \frac{K}{2} \sum_{i,j=1}^{3} \nabla_i u_j(\vec{x},t)\, \nabla_i u_j(\vec{x},t) - \frac{\Gamma}{2} \left(\vec{\nabla} \cdot \vec{u}(\vec{x},t) \right)^2 \right\} \tag{1}$$

where $u(\vec{x},t)$ is the displacement field, *i.e.* the local distortion of the continuous solid at a point $\vec{x}$ at time t.

1. Find the Classical Hamiltonian.
2. Quantize the system in (i) position and (ii) momentum space.
3. Derive expressions for the creation and annihilation operators of the normal modes.
4. Construct the following states:
   (a) The ground state $|0\rangle$.
   (b) A state with one longitudinal phonon with momentum $\vec{k}$.
   (c) A state with one longitudinal phonon of momentum $\vec{k}$ and one transverse phonon of momentum $\vec{q}$.

2 Creation and Annihilation Operators

1. Let $|\varphi\rangle$ be a one-particle state and $|\psi_1, \ldots, \psi_n\rangle$ an n-particle state (properly symmetrized or antisymmetrized). Show that the destruction operators $a(\varphi)$ satisfies

$$a(\varphi)|\psi_1, \ldots, \psi_n\rangle = \sum_{k=1}^{n} \zeta^{k-1} \langle \varphi | \psi_k \rangle |\psi_1, \ldots, \psi_{k-1}, \psi_{k+1}, \ldots, \psi_n\rangle \tag{2}$$

where $\zeta = \pm 1$ (+1 for Bosons and −1 for Fermions). Give an interpretation for this formula in the case in which , in a system of free fermions, the

n-particle state is the ground state $|G\rangle$ and $|\varphi\rangle$ is a single particle state localized at $\vec{x}$.

2. Prove the formula
$$\left[a(\varphi_1), a^\dagger(\varphi_2)\right]_{-\zeta} = \langle\varphi_1|\varphi_2\rangle \tag{3}$$
where
$$[A, B]_{-\zeta} = AB - \zeta BA \tag{4}$$

3 The Electron Gas

Consider an electron gas with one particle wave functions $\varphi_{\vec{p},\sigma}(\vec{x})$ for states with momentum $\vec{p}$ and z-component of the spin $\sigma =\uparrow, \downarrow$. The single particle energy of the state $\varphi_{\vec{p},\sigma}(\vec{x})$ is $E(\vec{p}) \equiv E(p^2)$ (independent of the spin and isotropic). Assume that the system has N particles of mass M, and that the energy function is
$$E(p^2) = \frac{\vec{p}^{\,2}}{2M} \tag{5}$$
The creation and annihilation operators for a fermion of spin $\sigma =\uparrow, \downarrow$ at a point $\vec{x}$ are denoted by $\psi^\dagger_\sigma(\vec{x})$ and $\psi_\sigma(\vec{x})$ and obey canonical anticommutation relations:
$$\begin{aligned} \{\psi_\sigma(\vec{x}), \psi_{\sigma'}(\vec{x}\,')\} &= \left\{\psi^\dagger_\sigma(\vec{x}), \psi^\dagger_{\sigma'}(\vec{x}\,')\right\} = 0, \\ \left\{\psi_\sigma(\vec{x}), \psi^\dagger_{\sigma'}(\vec{x}\,')\right\} &= \delta_{\sigma,\sigma'}\delta^{(3)}(\vec{x} - \vec{x}\,') \end{aligned} \tag{6}$$
and the second-quantized Hamiltonian is
$$H = \int d^3x \sum_{\sigma=\uparrow,\downarrow} \frac{\hbar^2}{2M}\, \vec{\nabla}\psi^\dagger_\sigma(\vec{x}) \cdot \vec{\nabla}\psi_\sigma(\vec{x}) \tag{7}$$

1. Construct the ground state $|G\rangle$ for a system with N particles. Find the values of the Fermi energy E_F and the Fermi momentum p_F. Calculate the ground state energy E_G.

2. Construct an excited state with one electron with spin $\uparrow$ and momentum $\vec{p}$ and one hole with spin $\downarrow$ and momentum $\vec{q}$, and find their energies.

3. The current carried by a one-particle state $|\varphi, \sigma\rangle$ is
$$\vec{j} = -i\frac{e\hbar}{2m} \int d^3x \sum_{\sigma=\uparrow,\downarrow} \left[\varphi^*_\sigma \vec{\nabla}\varphi_\sigma - \vec{\nabla}\varphi^*_\sigma \varphi_\sigma\right] \tag{8}$$
The corresponding second quantized *current operator* is
$$\vec{J} = -i\frac{e\hbar}{2m} \int d^3x \sum_{\sigma=\uparrow,\downarrow} \left[\psi^\dagger_\sigma(\vec{x})\vec{\nabla}\psi_\sigma(\vec{x}) - \vec{\nabla}\psi^\dagger_\sigma(\vec{x})\, \psi_\sigma(\vec{x})\right] \tag{9}$$
Check that this operator reproduces Eq.(8) by computing the matrix elements of the second-quantized operator of Eq.(9) between the one-particle states $|\varphi_\sigma\rangle = \psi^\dagger_\sigma(\varphi)|0\rangle$ and $|\chi_\sigma\rangle = \psi^\dagger_\sigma(\chi)|0\rangle$.

4. Find the Heisenberg equation of motion of the fermion operators $\psi_\sigma(\vec{x})$ and $\psi^\dagger_\sigma(\vec{x})$. Show that it has the same form as the Schródinger equation for the *wave function* of a free particle of mass M.
   Note: recall that the Heisenberg equation of motion of an operator $A(t)$ is
$$\frac{\partial A(t)}{\partial t} = \frac{i}{\hbar}\,[H, A(t)] \tag{10}$$
   where
$$A(t) = e^{iHt/\hbar} A e^{-iHt/\hbar} \tag{11}$$

5. Show that the particle density operator
$$\rho(\vec{x}, t) = \sum_\sigma \psi^\dagger_\sigma(\vec{x}, t)\psi_\sigma(\vec{x}, t) \tag{12}$$
   and the current density operator
$$\vec{J}(\vec{x}, t) = -i\frac{e\hbar}{2m} \sum_{\sigma=\uparrow,\downarrow} \left[\psi^\dagger_\sigma(\vec{x}, t)\vec{\nabla}\psi_\sigma(\vec{x}, t) - \vec{\nabla}\psi^\dagger_\sigma(\vec{x}, t)\ \psi_\sigma(\vec{x}, t)\right] \tag{13}$$
   obey the Heisenberg equation of motion
$$\frac{\partial \rho}{\partial t} + \vec{\nabla}\cdot\vec{J} = 0 \tag{14}$$
   i.e. the *continuity equation.*

4 Free fermions in one dimension

Consider a system of particles of mass M, charge e and spin spin-$\frac{1}{2}$. The particles are restricted to move on a line of length L and do not interact with each other. we will assume that the one-particle wave functions $\langle x, \sigma|\psi\rangle = \psi_\sigma(x)$ (with $\sigma = \uparrow, \downarrow$) obey periodic boundary conditions,
$$\psi(x) = \psi(x + L) \tag{15}$$
Consider the problem in general, without specifying the number of particles at first.

1. Write down the one-particle states $\psi_\sigma(x)$ which obey the boundary conditions given above.

2. Use fermion creation and annihilation operators $a^\dagger(x)$ and $a(x)$ in position space to write an expression for the Hamiltonian of this free fermion system in Fock space. Write the same Hamiltonian in momentum space.

3. Compute the anticommutators $\{a(p), a(p')\}$, $\{a^\dagger(p), a^\dagger(p')\}$ and $\{a(p), a^\dagger(p')\}$.

4. Construct the ground state $|gnd\rangle$ for a system of N fermions with spin spin-$\frac{1}{2}$. Assume that N is an even number and that $N/2$ is odd. Compute the Fermi energy E_F, namely the energy of the topmost occupied state. How many single particle states with this energy are present?.

5. Define a new set of creation and annihilation operators that annihilate the ground state $|gnd\rangle$. Write the Hamiltonian in terms of these new operators. Construct the spectrum of excitations in terms of particle and hole states with a given spin. Compute the excitation energies as a function of momentum p determine and the degeneracies of the single particle and single hole states with arbitrary spin.

6. Construct excited states with two particles, two holes and one particle and one hole. In each case assume that the excitations have momenta p and p' and arbitrary spin. Compute the energy and the total number of fermions for each state.

5 Thermodynamics of the Ideal Fermi Gas

In this problem you will work out the thermodynamic properties of an ideal spinless non-relativistic *Fermi* gas at finite temperature T and density $1/v$, where v is the specific volume.

1. Use the Grand Canonical Ensemble to show that the equation of state is given by

$$\frac{P}{k_B T} = \frac{1}{\lambda_T^3}\, f_{5/2}(z)$$
$$\frac{1}{v} = \frac{1}{\lambda_T^3}\, f_{3/2}(z)$$

where

$$f_{5/2}(z) = \frac{2}{\sqrt{\pi}} \int_0^\infty dx\, \sqrt{x}\, \log(1 + z\, e^{-x}) = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{z^n}{n^{5/2}}$$

$$f_{3/2}(z) = \frac{2}{\sqrt{\pi}} \int_0^\infty dx\, \sqrt{x}\, \frac{z e^{-x}}{1 + z\, e^{-x}} = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{z^n}{n^{3/2}}$$

and

$$\lambda_T = \left(\frac{2\pi\hbar^2}{mkT}\right)^{3/2}$$

is the thermal wavelength.

2. Show that in the low density (or high temperature) limit the equation of state free Fermi gas also has the form of a virial expansion. Compute the second virial coefficient for this free Fermi gas.

3. Show that the density $\rho = 1/v$ and the energy density $u = U/V$ can be written in the form

$$\begin{aligned} \rho &= \frac{1}{4\pi^2}\left(\frac{2m}{\hbar^2}\right)^{3/2}\int_0^\infty d\epsilon \frac{\sqrt{\epsilon}}{e^{\beta(\epsilon-\mu)}+1} \\ u &= \frac{1}{4\pi^2}\left(\frac{2m}{\hbar^2}\right)^{3/2}\int_0^\infty d\epsilon \frac{\epsilon^{3/2}}{e^{\beta(\epsilon-\mu)}+1} \end{aligned}$$

4. You will now use the integrals you just derived to investigate the low temperature limit of this gas. You will need to use the Sommerfeld expansion of these integrals:

$$\begin{aligned} I &= \int_0^\infty d\epsilon\, \frac{g(\epsilon)}{e^{\beta(\epsilon-\mu)}+1} \\ &= \int_0^\mu g(\epsilon)\, d\epsilon + \int_0^\infty \frac{g(\mu + x/\beta)}{e^x+1}\frac{dx}{\beta} - \int_0^{\beta\mu} \frac{g(\mu - x/\beta)}{e^x+1}\frac{dx}{\beta} \\ &\approx \int_0^\mu g(\epsilon)\, d\epsilon + \frac{\pi^2}{6\beta^2}\, g'(\mu) + \ldots \end{aligned} \tag{16}$$

Use this approximation to compute the specific heat of a Fermi gas at low temperatures. Write your answers in terms of the Fermi energy ϵ_F, the limiting value of the chemical potential at $T = 0$. Find the relation between ϵ_F and the specific volume v.

5. Use the same approximation to calculate the pressure in a Fermi gas at very low temperatures. What is the limiting value P_0 of the pressure at $T = 0$?