Physics 561 Condensed Matter Physics II
Homework 1-Solutions [1]

1 Second Quantization of an Elastic Solid

Consider a three-dimensional elastic solid in the continuum harmonic approximation. The classical Lagrangian for the system is

$$\mathcal{L} = \int d^3x \Bigg\{ \sum_{i=1}^{3} \frac{\rho}{2} \dot{u}_i^2(\vec{x,t}) - \frac{K}{2} \sum_{i,j=1}^{3} \nabla_i u_j(\vec{x},t) \nabla_i u_j(\vec{x},t) - \frac{\Gamma}{2} \left(\nabla \cdot \vec{u}(\vec{x},t)\right)^2 \Bigg\}$$

$\vec{u}(\vec{x},t)$ is the displacement field.

1. The canonically conjugate momenta are: $\Pi_i = \frac{\delta \mathcal{L}}{\delta \dot{u}_i} = \rho \dot{u}_i(\vec{x},t)$. From this the Hamiltonian can be easily found:

$$\begin{aligned} \mathcal{H} &= \int d^3x \sum_{i=1}^{3} \Pi_i \dot{u}_i - \mathcal{L} \\ &= \int d^3x \left\{ \frac{\vec{\Pi}^2}{2\rho} + \frac{K}{2} \nabla_i u_j(\vec{x},t) \nabla_i u_j(\vec{x},t) + \frac{\Gamma}{2} \left(\vec{\nabla} \cdot \vec{u}(\vec{x},t) \right)^2 \right\} \end{aligned} \tag{1}$$

2. (i) To quantize in real space, one promotes the fields to operators and imposes the canonical (equal time) commutation relations

$$\begin{aligned} [\hat{u}_i(\vec{x},t), \hat{u}_j(\vec{x},t)] &= [\hat{\Pi}_i(\vec{x},t), \hat{\Pi}_j(\vec{x},t)] = 0 \\ \left[\hat{u}_i(\vec{x},t), \hat{\Pi}_j(\vec{y},t)\right] &= i\hbar \delta_{ij} \delta(\vec{x} - \vec{y}) \end{aligned} \tag{2}$$

The Hamiltonian is regarded as an operator at a fixed time slice t.

$$\hat{H} = \int d^3x \left\{ \frac{\hat{\Pi}^2}{2\rho} + \frac{K}{2} \nabla_i \hat{u}_j(\vec{x}) \nabla_i \hat{u}_j(\vec{x}) + \frac{\Gamma}{2} \left(\vec{\nabla} \cdot \hat{u}(\vec{x}) \right)^2 \right\} \tag{3}$$

(ii) To quantize the Hamiltonian in momentum space, one needs to expand the operators in terms of the momentum basis. This amounts to a Fourier transform.

$$\begin{aligned} \tilde{u}_i(\vec{p}) = \sqrt{\rho} \int d^3x \, e^{i\vec{p}\cdot\vec{x}/\hbar} u_i(\vec{x}) \quad &; \quad \tilde{u}_i(\vec{x}) = \frac{1}{\sqrt{\rho}} \int \frac{d^3p}{(2\pi\hbar)^3} \, e^{-i\vec{p}\cdot\vec{x}/\hbar} u_i(\vec{p}) \\ \tilde{\Pi}_i(\vec{p}) = \frac{1}{\sqrt{\rho}} \int d^3x \, e^{i\vec{p}\cdot\vec{x}/\hbar} \Pi_i(\vec{x}) \quad &; \quad \tilde{\Pi}_i(\vec{x}) = \sqrt{\rho} \int \frac{d^3p}{(2\pi\hbar)^3} \, e^{-i\vec{p}\cdot\vec{x}/\hbar} \Pi_i(\vec{p}) \end{aligned} \tag{4}$$

In addition the displacement is a *real* quantity so one has $u(\vec{x})^\dagger = u(\vec{x}), \Pi(\vec{x})^\dagger = \Pi(\vec{x})$ which implies that

$$\tilde{u}_i^\dagger(\vec{p}) = \tilde{u}_i(-\vec{p}) \qquad \text{and} \qquad \tilde{\Pi}_i^\dagger(\vec{p}) = \tilde{\Pi}_i(-\vec{p}) \tag{5}$$

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The scale factors were choosen to keep the normalization of the canonical commutation relations and to get rid of prefactors so that the Hamiltonian will be the same form as the simple harmonic oscillator.

$$\begin{aligned}\left[\tilde{u}_i(\vec{p}), \tilde{\Pi}_i(\vec{p})\right] &= \int d^3x\, d^3y e^{-i\vec{p}\cdot\vec{x}/\hbar}\, e^{-i\vec{q}\cdot\vec{y}/\hbar}\, [u_i(\vec{x}), \Pi_j(\vec{y})] \\ &= i\hbar\delta_{ij} \int d^3x\, d^3y\ \delta(\vec{x}-\vec{y}) e^{-i\vec{p}\cdot\vec{x}/\hbar}\, e^{-i\vec{q}\cdot\vec{y}/\hbar} = i\hbar\delta_{ij}\int d^3x\, e^{i\vec{x}\cdot(\vec{p}+\vec{q})/\hbar} \\ &= i(2\pi\hbar)^3\, \delta_{ij}\, \delta(\vec{p}+\vec{q})\end{aligned} \tag{6}$$

All other commutation relations are vanishing which can be checked through a similar calculation. In momentum space, the Hamiltonian (1) can be written as

$$\mathcal{H} = \int \frac{d^3p}{(2\pi\hbar)^3}\left\{\frac{1}{2}\tilde{\Pi}(-\vec{p})\tilde{\Pi}(\vec{p}) + \frac{1}{2}\sum_{ij}\tilde{u}_i(-\vec{p})\left(\frac{K}{\rho}\vec{p}^2\delta_{ij} + \frac{\Gamma}{\rho}\vec{p}_i\vec{p}_j\right)\tilde{u}_j(\vec{p})\right\} \tag{7}$$

3. From (7), define

$$\omega_{ij}^2 = \left(\frac{K}{\rho}\vec{p}^2\delta_{ij} + \frac{\Gamma}{\rho}\vec{p}_i\vec{p}_j\right) \tag{8}$$

This is a 3×3 matrix and at this point is off diagonal. One would like to work in a basis where the Hamiltonian is diagonalized. For a 3×3 matrix, one expects three eigenvectors (normal modes). Observing that

$$\omega_{ij}^2 p_j = \left(\frac{K}{\rho}\vec{p}^2 + \frac{\Gamma}{\rho}\vec{p}_i\vec{p}_j\right)p_j = \left(\frac{K}{\rho}\vec{p}^2 + \frac{\Gamma}{\rho}\vec{p}^2\right)p_i$$

one immediately finds the first eigenvector, $\vec{p}/\left|p\right|$. Now, one can define two orthonormal vectors $\hat{e}^\alpha$ such that $\hat{\vec{e}}^\alpha\cdot\vec{p} = 0$ and $\hat{\vec{e}}^\alpha\cdot\hat{\vec{e}}^\beta = \delta_{\alpha\beta}$. Then its easy to see that the eigevnvalues are

$$\omega_{ij}^2\hat{\vec{e}}_j^\alpha = \left(\frac{K}{\rho}\vec{p}^2\delta_{ij} + \frac{\Gamma}{\rho}\vec{p}_i\vec{p}_j\right)\hat{e}_\alpha^j = \frac{K}{\rho}\vec{p}^2\hat{\vec{e}}_i^\alpha$$

One can then decompose $\tilde{u}_i(p)$ in the $\{\frac{\vec{p}}{|\vec{p}|^2}, \hat{\vec{e}}_1, \hat{\vec{e}}_2\}$ basis as $\tilde{u}_i(\vec{p}) = \tilde{u}_i^L(\vec{p}) + \tilde{u}_i^T(\vec{p})$.

$$\tilde{u}_i^L(\vec{p}) = \tilde{u}_L(\vec{p})\ \frac{\vec{p}}{|\vec{p}|} \qquad\qquad \tilde{u}_i^T(\vec{p}) = \sum_{\alpha=1,2}\tilde{u}_T^\alpha(\vec{p})\ \hat{\vec{e}}_\alpha \tag{9}$$

Inserting the expansion for $\tilde{u}_i(\vec{p})$ and $\tilde{\Pi}_i(\vec{p})$ one can find that the commutation relations are

$$\left[\tilde{u}_L(\vec{p}), \tilde{\Pi}_L(\vec{q})\right] = i(2\pi\hbar)^3\delta(\vec{p}+\vec{q}) \qquad \text{and} \qquad \left[\tilde{u}_T^\alpha(\vec{p}), \tilde{\Pi}_T^\beta(\vec{q})\right] = i(2\pi\hbar)^3\delta_{\alpha\beta}\delta(\vec{p}+\vec{q}) \tag{10}$$

Similarly, the momenta can be written this way as well. The Hamiltonian is then diagonal in this basis and can be written as $\mathcal{H} = \mathcal{H}_L + \mathcal{H}_T$ where

$$\begin{aligned}\mathcal{H}_L &= \frac{1}{2}\int\frac{d^3p}{(2\pi\hbar)^3}\left\{\tilde{\Pi}_L(-\vec{p})\tilde{\Pi}_L(\vec{p}) + \omega_L^2\tilde{u}_L(-\vec{p})\tilde{u}_L(\vec{p})\right\} \\ \mathcal{H}_T &= \frac{1}{2}\int\frac{d^3p}{(2\pi\hbar)^3}\sum_{\alpha=1,2}\left\{\tilde{\Pi}_T^\alpha(-\vec{p})\tilde{\Pi}_T^\alpha(\vec{p}) + \omega_T^{2\,\alpha}\,\tilde{u}_T^\alpha(-\vec{p})\tilde{u}_T^\alpha(\vec{p})\right\}\end{aligned} \tag{11}$$

where from before $\omega_L^2 = \left(\frac{K+\Gamma}{\rho}\right)$ and $\omega_T^{2\,\alpha} = \frac{K}{\rho}\vec{p}^2$. Both of these are of the form of the simple harmonic oscillator. By a similar set of manipulations, one can define the following creation and annihilation operators. Using (5) gives the creation operators.

$$\begin{aligned}
a_L(\vec{p}) &= \tfrac{1}{\sqrt{2\hbar}}\left(\sqrt{\omega_L(\vec{p})}\tilde{u}_L(\vec{p}) + \tfrac{i}{\sqrt{\omega_L(\vec{p})}}\tilde{\Pi}_L(\vec{p})\right)\\
a_L^\dagger(\vec{p}) &= \tfrac{1}{\sqrt{2\hbar}}\left(\sqrt{\omega_L(\vec{p})}\tilde{u}_L(-\vec{p}) - \tfrac{i}{\sqrt{\omega_L(\vec{p})}}\tilde{\Pi}_L(-\vec{p})\right)\\
a_T^\alpha(\vec{p}) &= \tfrac{1}{\sqrt{2\hbar}}\left(\sqrt{\omega_T^\alpha(\vec{p})}\tilde{u}_T^\alpha(\vec{p}) + \tfrac{i}{\sqrt{\omega_L(\vec{p})}}\tilde{\Pi}_L(\vec{p})\right)\\
a_T^{\alpha\,\dagger}(\vec{p}) &= \tfrac{1}{\sqrt{2\hbar}}\left(\sqrt{\omega_T^\alpha(\vec{p})}\tilde{u}_T^\alpha(-\vec{p}) - \tfrac{i}{\sqrt{\omega_L(\vec{p})}}\tilde{\Pi}_L(-\vec{p})\right)
\end{aligned} \tag{12}$$

Inserting into (10) yields the commutation relations

$$\begin{aligned}
[a_L(\vec{p}), a_L(\vec{q})] &= \left[a_L^\dagger(\vec{p}), a_L^\dagger(\vec{q})\right] = 0\\
\left[a_T^\alpha(\vec{p}), a_T^\beta(\vec{q})\right] &= \left[a_T^{\alpha\,\dagger}(\vec{p}), a_T^{\beta\,\dagger}(\vec{q})\right] = 0\\
[a_T^\alpha(\vec{p}), a_L(\vec{q})] &= \left[a_T^\alpha(\vec{p}), a_L^\dagger(\vec{q})\right] = 0\\
\left[a_L(\vec{p}), a_L^\dagger(\vec{q})\right] &= (2\pi)^3\delta^3(\vec{p}+\vec{q})\\
\left[a_T^\alpha(\vec{p}), a_T^{\beta\,\dagger}(\vec{q})\right] &= (2\pi)^3\delta_{\alpha\beta}\delta^3(\vec{p}+\vec{q})
\end{aligned} \tag{13}$$

4. One can invert (12) and find that

$$\begin{aligned}
\tilde{u}_L(\vec{p}) &= \sqrt{\frac{\hbar}{2\omega_L(\vec{p})}}\left(a_L(\vec{p}) + a_L^\dagger(-\vec{p})\right) \qquad \tilde{u}_T^\alpha(\vec{p}) = \sqrt{\frac{\hbar}{2\omega_T^\alpha(\vec{p})}}\left(a_T^\alpha(\vec{p}) + a_T^{\alpha\,\dagger}(-\vec{p})\right)\\
\tilde{\Pi}_L(\vec{p}) &= -i\sqrt{\frac{\hbar\omega_L(\vec{p})}{2}}\left(a_L(\vec{p}) - a_L^\dagger(-\vec{p})\right) \qquad \tilde{\Pi}_T^\alpha(\vec{p}) = -i\sqrt{\frac{\hbar\omega_T^\alpha(\vec{p})}{2}}\left(a_L^\alpha(\vec{p}) - a_T^{\alpha\,\dagger}(-\vec{p})\right)
\end{aligned} \tag{14}$$

Using these relations in the momentum space representation of the Hamiltonian, one can obtain the expected harmonic oscillator form

$$\mathcal{H} = \int \frac{d^3p}{(2\pi\hbar)^3}\left(\hbar\omega_L(a_L^\dagger(\vec{p})a_L(\vec{p}) + \frac{1}{2}(2\pi\hbar)^3\delta^3(0)) + \sum_{\alpha=1,2}\hbar\omega_T^\alpha(a_T^{\alpha\,\dagger}(\vec{p})a_T^\alpha(\vec{p}) + \frac{1}{2}(2\pi\hbar)^3\delta^3(0))\right) \tag{15}$$

(a) By definition, the ground state is the state annihilated by $a_L(\vec{p})$ and $a_T^\alpha(\vec{p})$.

$$a_L(\vec{p})\,|gnd\rangle = 0 \qquad \text{and} \qquad a_T^\alpha(\vec{p})\,|gnd\rangle = 0 \tag{16}$$

(b) To create a longitudinal mode with momentum $\vec{k}$

$$a_L^\dagger(\vec{k})\,|gnd\rangle \tag{17}$$

(c) A mode with one longitudinal mode with momentum $\vec{k}$ and one transverse mode with momentum $\vec{q}$ (with two possible polarizations $\alpha = 1, 2$).

$$a_L^\dagger(\vec{k})a_T^{\dagger\,\alpha}(\vec{q})\,|gnd\rangle \tag{18}$$

2 Creation and Annihilation Operators

1. Let $|\psi_1 \ldots . \psi_n\rangle$ be an n particle state and $|\chi_1 \ldots . \chi_{n-1}\rangle$ be an $n-1$ particle state. The creation operator acts on the $n-1$ particle state as mentioned in the lecture notes (c.f. Fetter, Chapter 1 also): $\hat{a}^\dagger(\phi)\,|\chi_1 \ldots . \chi_{n-1}\rangle = |\phi, \chi_1 \ldots . \chi_{n-1}\rangle$. Then, observe that

$$\langle \chi_1 \ldots . \chi_{n-1}|\,\hat{a}(\phi)\,|\psi_1 \ldots . \psi_n\rangle = \left(\langle \psi_1 \ldots . \psi_n|\,\hat{a}^\dagger(\phi)\,|\chi_1 \ldots . \chi_{n-1}\rangle\right)^* = \left(\langle \psi_1 \ldots . \psi_n|\phi, \chi_1 \ldots . \chi_{n-1}\rangle\right)^*$$

The overlap can be written as a determinant or permanent.

$$\langle \psi_1 \ldots . \psi_n|\phi, \chi_1 \ldots . \chi_{n-1}\rangle = \begin{vmatrix} \langle\psi_1|\phi\rangle & \ldots . & \langle\psi_1|\chi_{n-1}\rangle \\ . & & . \\ . & & . \\ . & & . \\ \langle\psi_n|\phi\rangle & \ldots . & \langle\psi_n|\chi_{n-1}\rangle \end{vmatrix}_\zeta$$

where if $\zeta = +1$ its a permanent and if $\zeta = -1$ its a determinant. The determinant/permanent can be expanded along the first column.

$$\begin{aligned}\langle \psi_1 \ldots . \psi_n|\phi, \chi_1 \ldots . \chi_{n-1}\rangle &= \sum_{k=1}^{n} \zeta^{k-1}\,\langle\psi_k|\phi\rangle \begin{vmatrix} \langle\psi_1|\chi_1\rangle & \ldots . & \langle\psi_1|\chi_{n-1}\rangle \\ . & & . \\ . & (\text{no } \psi_k) & . \\ . & & . \\ \langle\psi_n|\chi_1\rangle & \ldots . & \langle\psi_n|\chi_{n-1}\rangle \end{vmatrix}_\zeta \\ &= \sum_{k=1}^{n} \zeta^{k-1}\,\langle\psi_k|\phi\rangle\,\langle\psi_1 \ldots \psi'_k \ldots \psi_n|\chi_1 \ldots \chi_{n-1}\rangle\end{aligned}$$

where ψ'_k means that ψ_k is omitted. One really wants the conjugate of the above to find the action of the destruction operator.

$$\langle \chi_1 \ldots . \chi_{n-1}|\,\hat{a}(\phi)\,|\psi_1 \ldots . \psi_n\rangle = \sum_{k=1}^{n} \zeta^{k-1}\,\langle\phi|\psi_k\rangle\,\langle\chi_1 \ldots \chi_{n-1}|\psi_1 \ldots \psi'_k \ldots \psi_n\rangle$$

The state $|\chi_1 \ldots \chi_{n-1}\rangle$ is an arbitrary $n-1$ particle state so that one can conclude that the action of the annihilation operator is

$$\hat{a}(\phi)\,|\psi_1 \ldots . \psi_n\rangle = \sum_{k=1}^{n} \zeta^{k-1}\,\langle\phi|\psi_k\rangle\,|\psi_1 \ldots \psi'_k \ldots \psi_n\rangle \tag{19}$$

Now, we wish to consider the case where the n particle state is the filled Fermi sea $|gnd\rangle$ and the state $|\phi\rangle$ is a state localized at $\vec{x}$, *i.e.* $|\phi\rangle = |\vec{x}\rangle$. From the equation above, the action of the annihilation operator is

$$\hat{a}(\vec{x})\,|\psi_1 \ldots . \psi_n\rangle = \sum_{k=1}^{n} \zeta^{k-1}\,\langle\vec{x}|\psi_k\rangle\,|\psi_1 \ldots \psi'_k \ldots \psi_n\rangle = \sum_{k=1}^{n} \zeta^{k-1} e^{i\vec{p}_k\cdot\vec{x}}\,|\psi_1 \ldots \psi'_k \ldots \psi_n\rangle \tag{20}$$

2. Now that we know the action of the annihilation operator on the n-particle state, its easy to evaluate the commutation, $[\hat{a}(\phi_1), \hat{a}^\dagger(\phi_2)]_\zeta$.

$$\begin{aligned}[\hat{a}(\phi_1), \hat{a}^\dagger(\phi_2)]_\zeta\,|\psi_1 \ldots \psi_n\rangle &= \left(\hat{a}(\phi_1)\hat{a}^\dagger(\phi_2) - \zeta\hat{a}^\dagger(\phi_2)\hat{a}(\phi_1)\right)|\psi_1 \ldots \psi_n\rangle \\ &= \hat{a}(\phi_1)\,|\phi_2, \psi_1 \ldots \psi_n\rangle - \zeta\hat{a}^\dagger(\phi_2)\sum_{k=1}^{n} \zeta^{k-1}\,\langle\phi_1|\psi_k\rangle\,|\psi_1 \ldots \psi'_k \ldots \psi_n\rangle \\ &= \sum_{k=1}^{n+1} \zeta^{k-1}\,\langle\phi_1|\psi_k\rangle\,|\phi_2,\psi_1 \ldots \psi'_k \ldots \psi_n\rangle - \zeta\sum_{k=1}^{n} \zeta^{k-1}\,\langle\phi_1|\psi_k\rangle\,|\phi_2, \psi_1 \ldots \psi'_k \ldots \psi_n\rangle\end{aligned}$$

The sum above include ϕ_2 in the $(n+1)^{th}$ position. This first sum can be split into the piece involving ϕ_2 and the n particle state.

$$\begin{aligned}[\hat{a}(\phi_1), \hat{a}^\dagger(\phi_2)]_\zeta \left|\psi_1 \ldots \psi_n\right\rangle &= \left\langle\phi_1|\phi_2\right\rangle \left|\psi_1 \,..\psi_k' \,..\psi_n\right\rangle + \sum_{k=1}^{n} \zeta^k \left\langle\phi_1|\psi_k\right\rangle \left|\phi_2, \psi_1 \,..\psi_k' \,..\psi_n\right\rangle \\ &- \zeta \sum_{k=1}^{n} \zeta^{k-1} \left\langle\phi_1|\psi_k\right\rangle \left|\phi_2, \psi_1 \,..\psi_k' \,..\psi_n\right\rangle \\ &= \left\langle\phi_1|\phi_2\right\rangle \left|\psi_1 \,..\psi_k' \,..\psi_n\right\rangle\end{aligned}$$

Hence, the action of $[\hat{a}(\phi_1), \hat{a}^\dagger(\phi_2)]_\zeta$ on an arbitrary state is simply

$$[\hat{a}(\phi_1), \hat{a}^\dagger(\phi_2)]_\zeta = \left\langle\phi_1|\phi_2\right\rangle \tag{21}$$

3 The Electron Gas

Consider an electron gas with one particle wave functions $\phi_{\vec{p},\sigma}(\vec{x})$ with momentum $\vec{p}$ and z-component spin $\sigma =\uparrow,\downarrow$. The single particle energy of the state is given by $E(p^2) = \frac{\vec{p}^2}{2M}$. The creation and annihilation operators, $\psi_\sigma^\dagger(\vec{x}), \psi_\sigma(\vec{x})$ obey the usual anti-commutation relations, $\left\{\psi_\sigma(\vec{x}), \psi_{\sigma'}^\dagger(\vec{y})\right\} = \delta_{\sigma,\sigma'}\delta^3(\vec{x}-\vec{y})$. All other anti-commutation relations are vanishing. The second quantized Hamiltonian is given by

$$\mathcal{H} = \int d^3x \sum_{\sigma=\uparrow,\downarrow} \frac{\hbar^2}{2M} \nabla\psi_\sigma^\dagger(\vec{x}) \cdot \nabla\psi_\sigma(\vec{x})$$

1. In momentum space, one can define the number operator $\hat{N} = \int \frac{d^3p}{(2\pi)^3} \sum_\sigma \hat{\psi}_{\vec{p},\sigma}^\dagger \hat{\psi}_{\vec{p},\sigma}$. Defining the operators $b_{\vec{p},\sigma} = \hat{\psi}_{\vec{p},\sigma}^\dagger$ for $|\vec{p}| \leq p_F$ and $a_{\vec{p},\sigma} = \hat{\psi}_{\vec{p},\sigma}$ for $|\vec{p}| \geq p_F$. Performing the particle-hole transformation, the number operator can be written as

$$\hat{N} = \int \frac{d^3p}{(2\pi\hbar)^3} \sum_\sigma \left\{\theta(p_F - |\vec{p}|)\hat{a}_{\vec{p},\sigma}^\dagger \hat{a}_{\vec{p},\sigma} + \theta(|\vec{p}| - p_F)\hat{b}_{\vec{p},\sigma}\hat{b}_{\vec{p},\sigma}^\dagger\right\}$$

which can be normal ordered to give,

$$\begin{aligned}\hat{N} &= \int \frac{d^3p}{(2\pi\hbar)^3} \sum_\sigma \left\{\theta(p_F - |\vec{p}|)\hat{a}_{\vec{p},\sigma}^\dagger \hat{a}_{\vec{p},\sigma} - \theta(|\vec{p}| - p_F)\hat{b}_{\vec{p},\sigma}^\dagger \hat{b}_{\vec{p},\sigma} + \theta(p_F - |p|)\delta^3(0)\right\} \\ &=: \hat{N} : + \int \frac{d^3p}{(2\pi\hbar)^3} \sum_\sigma \theta(p_F - |p|)\delta^3(0)\end{aligned} \tag{22}$$

The divergent (in the thermodynamic limit) second term represents the ground state occupation number. Evaluating it,

$$N_{gnd} = \int \frac{d^3p}{(2\pi\hbar)^3} \sum_{\sigma=\uparrow,\downarrow} \theta(p_F - |p|)\delta^3(0) = \frac{V}{(2\pi\hbar)^3} \int_0^{p_F} d\,|\vec{p}|\; 4\pi\,|\vec{p}|^2 \sum_{\sigma=\uparrow,\downarrow} 1 = \frac{2}{3}\frac{V}{2\pi^2\hbar^3} p_F^3 \tag{23}$$

Letting $n = N/V$ the number density, the Fermi momentum can be found by inverting the above expression. This also leads directly to the Fermi energy.

$$p_F = \hbar(3\pi^2 n)^{1/3} \qquad \text{and} \qquad E_F = \frac{1}{2M}p_F^2 = \frac{\hbar^2}{2M}(3\pi^2 n)^{2/3} \tag{24}$$

To find the ground state energy, one can proceed as before (c.f. Problem 1) and replace the real space operators $\psi_\sigma(\vec{x})$ with their momentum space counter parts $\hat{\psi}_{\vec{p},\sigma}$ and then perform a particle-hole transformation as in the previous subpart. The ground state energy is then

$$E_{gnd} = \frac{V}{(2\pi\hbar)^3} \int d^3p \sum_\sigma \theta(p_F - |\vec{p}|)\delta^3(0)\frac{|\vec{p}|^2}{2M} = \frac{V}{2\pi^2\hbar^3}\int_0^{p_F} d\,|\vec{p}| \sum_\sigma \frac{|p|^4}{2M} = \frac{V}{2\pi^2}\frac{p_F^5}{5M} \tag{25}$$

2. The excited state with one spin-$\uparrow$ electron and momentum $\vec{k}$ is by definition, $|\uparrow,\vec{k}\rangle = \hat{a}^\dagger_{\vec{k},\uparrow}|Gnd\rangle$ where $|\vec{k}| > p_F$. Using the canonical commutation relations $\{\hat{a}_{\vec{p},\sigma}, \hat{a}^\dagger_{\vec{q},\sigma'}\} = (2\pi\hbar)^3\delta^3(\vec{p}-\vec{q})\delta_{\sigma,\sigma'}$ and $\{\hat{b}_{\vec{p},\sigma}, \hat{a}^\dagger_{\vec{q},\sigma'}\} = 0$,

$$\begin{aligned}
\hat{H}|\uparrow,\vec{k}\rangle &= \int \frac{d^3p}{(2\pi\hbar)^3} E(\vec{p}) \sum_\sigma \left\{\theta(p_F - |\vec{p}|)\hat{a}^\dagger_{\vec{p},\sigma}\hat{a}_{\vec{p},\sigma} - \theta(|\vec{p}| - p_F)\hat{b}^\dagger_{\vec{p},\sigma}\hat{b}_{\vec{p},\sigma}\right\}|\uparrow,\vec{k}\rangle + E_{gnd}|\uparrow,\vec{k}\rangle \\
&= \int \frac{d^3p}{(2\pi\hbar)^3} E(\vec{p}) \sum_\sigma \left\{\theta(p_F - |\vec{p}|)\hat{a}^\dagger_{\vec{p},\sigma}\hat{a}_{\vec{p},\sigma}\hat{a}^\dagger_{\vec{k},\uparrow}|Gnd\rangle\right\} + E_{gnd}|\uparrow,\vec{k}\rangle \\
&= \int \frac{d^3p}{(2\pi\hbar)^3} E(\vec{p}) \sum_\sigma \left\{\theta(p_F - |\vec{p}|)(2\pi\hbar)^3\delta^3(\vec{p}-\vec{k})\delta_{\sigma,\uparrow}\ \hat{a}^\dagger_{\vec{p},\sigma}|Gnd\rangle\right\} + E_{gnd}|\uparrow,\vec{k}\rangle \\
&= (E(\vec{k}) + E_{gnd})|\uparrow,\vec{k}\rangle
\end{aligned} \tag{26}$$

The energy of the state $|\uparrow,\vec{k}\rangle$ *relative* to the ground state energy is simply $E(\vec{k}) = \frac{|\vec{k}|^2}{2M}$ as expected. A similar set of manipulations can be done for the *hole* state $|\downarrow,\vec{q}\rangle = \hat{b}^\dagger_{\vec{q},\downarrow}|Gnd\rangle$ with $|\vec{q}| < p_F$ with the result $E(\vec{q}) = -\frac{|\vec{q}|^2}{2M}$. A state composed of one hole with spin $\downarrow$ and one electron with spin $\uparrow$ can be constructed similarly, $\hat{a}^\dagger_{\vec{k},\uparrow}\hat{b}^\dagger_{\vec{q},\downarrow}|Gnd\rangle$. The hole and electron contributions to te energy are additive, $E = -\frac{|\vec{q}|^2}{2M} + \frac{|\vec{k}|^2}{2M}$

3. In second quantized notation, the current operator is given by

$$\vec{J} = -i\frac{e\hbar}{2m}\int d^3x \sum_\sigma \left[\psi^\dagger_\sigma(\vec{x})\nabla\psi_\sigma(\vec{x}) - \left(\nabla\psi^\dagger_\sigma(\vec{x})\right)\psi_\sigma(\vec{x})\right]$$

Here, we are asked to find its reprsentation in the single-particle states basis so we need to compute the overlap between one-particle states $|\phi_\sigma\rangle = \psi^\dagger_\sigma(\phi)|0\rangle$ and $|\chi_\sigma\rangle = \psi^\dagger_\sigma(\chi)|0\rangle$. Hence,

$$\begin{aligned}
\langle\phi_\sigma|\vec{J}|\chi_\sigma\rangle &= \int d^3y d^3z\ \langle\phi_\sigma|\vec{y}\rangle\langle\vec{y}|\vec{J}|\vec{z}\rangle\langle\vec{z}|\chi_\sigma\rangle \\
&= \int d^3y d^3z\, \phi^*_\sigma(\vec{y})\langle\vec{y}|\vec{J}|\vec{z}\rangle\chi_\sigma(\vec{z}) \\
&= \int d^3y d^3z\, \phi^*_\sigma(\vec{y})\langle 0|\psi_\sigma(\vec{y})\vec{J}\psi^\dagger_{\sigma'}(\vec{z})|0\rangle\chi_\sigma(\vec{z})
\end{aligned}$$

A complete set of position states was inserted in the first line. Now the quantity, $\langle 0|\psi^\dagger_\sigma(\vec{y})\vec{J}\psi_{\sigma'}(\vec{z})|0\rangle$ can be simplified using the anti-commutation relations for fermions

$$\begin{aligned}
\langle 0|\psi^\dagger_\sigma(\vec{y})\vec{J}\psi_{\sigma'}(\vec{z})|0\rangle &= -i\frac{e\hbar}{2m}\int d^3x\,\langle 0|\psi_\sigma(\vec{y})\left[\sum_\sigma \psi^\dagger_\sigma(\vec{x})\nabla\psi_\sigma(\vec{x}) - \left(\nabla\psi^\dagger_\sigma(\vec{x})\right)\psi_\sigma(\vec{x})\right]\psi^\dagger_{\sigma'}(\vec{z})|0\rangle \\
&= -i\frac{e\hbar}{2m}\sum_\sigma\int d^3x\,\langle 0|\left[\delta(\vec{x}-\vec{y})\nabla\delta(\vec{x}-\vec{z}) - \psi_\sigma(\vec{y})\left(\nabla\psi^\dagger_\sigma(\vec{x})\right)\delta(\vec{x}-\vec{z})\right]|0\rangle \qquad (27) \\
&= -i\frac{e\hbar}{2m}\sum_\sigma\int d^3x\,\langle 0|\left[\delta(\vec{x}-\vec{y})\nabla\delta(\vec{x}-\vec{z}) - \left(\nabla\delta(\vec{x}-\vec{y})\right)\delta(\vec{x}-\vec{z})\right]|0\rangle
\end{aligned}$$

The term in $[\ldots]$ is a now a c-number and $1 = \langle 0|0\rangle$. Inserting this result into the previous, one finds

$$\begin{aligned}
\langle\phi_\sigma|\,\vec{J}\,|\chi_\sigma\rangle &= -i\frac{e\hbar}{2m}\sum_\sigma \int d^3x\, d^3y\, d^3z\ \phi^*_\sigma(\vec{y})\left(\delta(\vec{x}-\vec{y})\nabla\delta(\vec{x}-\vec{z}) - \big(\nabla\delta(\vec{x}-\vec{y})\big)\delta(\vec{x}-\vec{z})\right)\chi_\sigma(\vec{z})\\
&= -i\frac{e\hbar}{2m}\sum_\sigma \int d^3x\, d^3z\ \phi^*_\sigma(\vec{x})\nabla\delta(\vec{x}-\vec{z})\chi_\sigma(\vec{z}) - \int d^3x\, d^3y\ \phi^*_\sigma(\vec{y})\big(\nabla\delta(\vec{x}-\vec{y})\big)\chi_\sigma(\vec{x})\\
&= -i\frac{e\hbar}{2m}\sum_\sigma \int d^3x\, d^3z\ \left(-(\nabla\phi^*_\sigma(\vec{x}))\delta(\vec{x}-\vec{z})\chi_\sigma(\vec{z}) + \int d^3x\, d^3y\ \phi^*_\sigma(\vec{y})\delta(\vec{x}-\vec{y})(\nabla\chi_\sigma(\vec{x}))\right)\\
&= -i\frac{e\hbar}{2m}\sum_\sigma \int d^3x\ \left(\phi^*_\sigma(\vec{x})\nabla\chi_\sigma(\vec{x}) - \big(\nabla\phi^*_\sigma(\vec{x})\big)\chi_\sigma(\vec{x})\right)
\end{aligned}$$

Looking at the diagonal components, $\phi_\sigma(\vec{x}) = \chi_\sigma(\vec{x})$, one obtains the desired result.

$$\vec{j} = -i\frac{e\hbar}{2m}\int d^3x \sum_\sigma \left(\phi^*_\sigma(\vec{x})\nabla\phi_\sigma(\vec{x}) - \big(\nabla\phi^*_\sigma(\vec{x})\big)\phi_\sigma(\vec{x})\right) \tag{28}$$

4. Recall that in the Heisenberg representation, the operators depend on time and their time evolution is governed by

$$-i\hbar\frac{\partial}{\partial t}\hat{O}(\vec{x},t) = [\hat{H}, O(\vec{x,t})] \qquad \text{where} \qquad \hat{O}(x,t) = e^{-i\hat{H}t}\hat{O}(\vec{x})e^{i\hat{H}t}$$

For fermion operator $\psi_\sigma(\vec{x})$, one can follow this prescription to find the equations of motion. First, its useful to evaluate the commutator,

$$\begin{aligned}
\left[\hat{H}, \psi_{\sigma'}(\vec{y})\right] &= \int d^3x\, \frac{\hbar^2}{2M}\sum_{\sigma=\uparrow,\downarrow}\left[\nabla\psi^\dagger_\sigma(\vec{x})\cdot\nabla\psi_\sigma(\vec{x}), \psi_{\sigma'}(\vec{y})\right]\\
&= \int d^3x\, \frac{\hbar^2}{2M}\sum_\sigma \left(\nabla\psi^\dagger_\sigma(\vec{x})\cdot\nabla\psi_\sigma(\vec{x})\psi_{\sigma'}(\vec{y}) - \psi_{\sigma'}(\vec{y})\nabla\psi^\dagger_\sigma(\vec{x})\cdot\nabla\psi_\sigma(\vec{x})\right)\\
&= \int d^3x\, \frac{\hbar^2}{2M}\sum_\sigma \left(-(\nabla\,\delta_{\sigma,\sigma'}\,\delta^3(\vec{x}-\vec{y}))\ \cdot\nabla\psi_\sigma(\vec{x})\right)\\
&= \frac{\hbar^2}{2M}\nabla\cdot\nabla\psi_{\sigma'}(\vec{y})\\
&= \frac{\hbar^2}{2M}\nabla^2\psi_{\sigma'}(\vec{y})
\end{aligned} \tag{29}$$

Then it is straight forward to see that

$$\left[\hat{H}, \psi_{\sigma'}(\vec{y},t)\right] = \left[\hat{H}, e^{-i\hat{H}t}\psi_{\sigma'}(\vec{y})e^{i\hat{H}t}\right] = e^{-i\hat{H}t}\left[\hat{H}, \psi_{\sigma'}(\vec{y})\right]e^{i\hat{H}t} = \frac{\hbar^2}{2M}\nabla^2\psi_{\sigma'}(\vec{y},t) \tag{30}$$

Indeed, putting it all together, one arrives at the Schrödinger equation.

$$i\hbar\partial_t\psi_{\sigma'}(\vec{y},t) = -\frac{\hbar^2}{2M}\nabla^2\psi_{\sigma'}(\vec{y},t) \tag{31}$$

For the adjoint, a similar procedure can be followed. The result is

$$i\hbar\partial_t\psi^\dagger_{\sigma'}(\vec{y},t) = \frac{\hbar^2}{2M}\nabla^2\psi^\dagger_{\sigma'}(\vec{y},t) \tag{32}$$

5. Using (31) and (32) the time derivative of the density operator is easily evaluated.

$$\begin{aligned}
\frac{\partial}{\partial t}\rho(\vec{x},t) &= \sum_\sigma \left\{\dot{\psi}^\dagger_\sigma(\vec{x},t)\psi_\sigma(\vec{x},t) + \psi^\dagger_\sigma(\vec{x},t)\dot{\psi}_\sigma(\vec{x},t)\right\}\\
&= -\frac{i\hbar}{2M}\sum_\sigma (\nabla^2\psi^\dagger_\sigma(\vec{x},t))\psi_\sigma(\vec{x},t) + \psi^\dagger_\sigma(\vec{x},t)(\nabla^2\psi_\sigma(\vec{x},t))
\end{aligned} \tag{33}$$

Explicitly taking the divergence of the current and performing one integration by parts, one finds

$$\begin{aligned}\nabla \cdot \vec{J} &= -\frac{i\hbar}{2M} \int d^3x \sum_\sigma \nabla \cdot \left(\psi^\dagger_\sigma(\vec{x}) \nabla \psi_\sigma(\vec{x}) - \left(\nabla \psi^\dagger_\sigma(\vec{x}) \right) \psi_\sigma(\vec{x}) \right) \\ &= \frac{i\hbar}{2M} \int d^3x \sum_\sigma (\nabla^2 \psi^\dagger_\sigma(\vec{x},t)) \psi_\sigma(\vec{x},t) + \psi^\dagger_\sigma(\vec{x},t) (\nabla^2 \psi_\sigma(\vec{x},t)) \end{aligned} \tag{34}$$

Its easy to verify that the current and density operators satisfy the continuity equation.

$$\frac{\partial}{\partial t} \rho(\vec{x},t) + \nabla \cdot \vec{J} = 0 \tag{35}$$

4 Free Fermions in One Dimension

Consider a system of non-interacting particles with mass M, charge e and spin-$\frac{1}{2}$ living on a line of length L. Suppose that the one particle wave functions obey periodic boundary conditions $\psi(x) = \psi(x+L)$.

1. With n an integer (positive and negative), the (normalized!) eigenfunctions and their energies are:

$$\psi(x) = A \frac{1}{\sqrt{L}} e^{i \frac{2\pi}{L} n} \qquad ; \qquad E_n = \frac{2\hbar^2 \pi^2}{mL^2} n^2 \tag{36}$$

2. In position space, the Hamiltonian $\hat{H} = \frac{\hat{p}^2}{2m}$ takes the usual form

$$\langle x, \sigma | \hat{H} | y, \sigma' \rangle = -\frac{\hbar^2}{2m} \nabla^2 \delta_{\sigma,\sigma'} \delta(x-y)$$

In terms of fermionic creation and annihilation operators

$$\hat{H} = \sum_\sigma \int dx \; \hat{a}^\dagger_\sigma(x) \left(-\frac{\hbar^2}{2m} \nabla^2 \right) \hat{a}_\sigma(x) \tag{37}$$

The creation and annihilation operators satisfy the anti-commutation relations $\{a_{\sigma'}(x), a^\dagger_\sigma(y)\} = \delta_{\sigma,\sigma'} \delta(x-y)$. One can now define the discrete Fourier transforms

$$\hat{a}_\sigma(x) = \sum_{n=-\infty}^{\infty} \frac{1}{\sqrt{L}} e^{i \frac{2\pi}{L} n} \hat{a}_{\sigma,n} \qquad ; \qquad \hat{a}^\dagger_\sigma(x) = \sum_{n=-\infty}^{\infty} \frac{1}{\sqrt{L}} e^{-i \frac{2\pi}{L} n} \hat{a}^\dagger_{\sigma,n}$$

Inserting for the position space operators in terms of their Fourier transformed counter parts, one finds the Hamiltonian

$$\hat{H} = \sum_{n,\sigma} E_n \; a^\dagger_{\sigma,n} \hat{a}_{\sigma,n} \tag{38}$$

3. Inverting the Fourier transform pair,

$$\hat{a}_{\sigma,n} = \frac{1}{\sqrt{L}} \int dx \; e^{i \frac{2\pi}{L} n} \hat{a}_\sigma(x) \qquad ; \qquad \hat{a}^\dagger_{\sigma,n} = \frac{1}{\sqrt{L}} \int dx \; e^{-i \frac{2\pi}{L} n} \hat{a}_\sigma(x)$$

The commutation relations for the momentum space creation and annihilation operators are straight forwardly found.

$$\{\hat{a}_{\sigma,n}, \hat{a}^\dagger_{\sigma',m}\} = \frac{1}{L} \int dxdy \; e^{i \frac{2\pi}{L} nx} e^{-i \frac{2\pi}{L} my} \{\hat{a}_\sigma(x), \hat{a}^\dagger_{\sigma'}(y)\} = \frac{1}{L} \int dxdy \; e^{i \frac{2\pi}{L} nx} e^{-i \frac{2\pi}{L} my} \delta_{\sigma,\sigma'} \delta(x-y)$$

$$\{\hat{a}_{\sigma,n}, \hat{a}^\dagger_{\sigma',m}\} = \delta_{\sigma,\sigma'} \delta_{nm} \tag{39}$$

The rest of the commutation relations can be similarly found. $\{\hat{a}_{\sigma,n}, \hat{a}_{\sigma',m}\} = \{\hat{a}^\dagger_{\sigma,n}, \hat{a}^\dagger_{\sigma',m}\} = 0.$

4. Supposing that there are N even number of fermions and given that each energy level is four fold degenerate (because of the spin degree of freedom and invariance under $n \to -n$), the ground state is the one where $N/2$ fermions occupy all the spin down states and $N/2$ fermions occupy the spin up states. With $N/2$ fermions of one type, one can fill $n = (N/2-1)/2$ positive states and $n = (N/2-1)/2$ negative states. The ground state is then

$$|gnd\rangle = \prod_{\sigma=\uparrow,\downarrow} \prod_{n=\frac{-(N-2)}{4}}^{\frac{(N-2)}{4}} \hat{a}^\dagger_{\sigma,n} |0\rangle \tag{40}$$

The Fermi energy E_F is simply the energy of the top-most filled level. This corresponds to $n = \pm\frac{(N-2)}{4}$. Along with the spin degeneracy (2-fold) and the degeneracy in $\pm n$ (2-fold), the top most level is 4-fold degenerate.

$$E_F = \frac{2\hbar^2\pi^2}{mL^2}\left(\frac{N-2}{4}\right)^2 \tag{41}$$

5. Defining $n_F = \frac{(N-2)}{4}$ one can define hole and electron annihilation operators as

$$\begin{aligned} \hat{b}_{\sigma,n} &= \hat{a}^\dagger_{\sigma,n} \quad \text{for} \quad |n| \leq n_F \\ \hat{a}_{\sigma,n} &= \hat{a}_{\sigma,n} \quad \text{for} \quad |n| > n_F \end{aligned} \tag{42}$$

The Hamiltonian can then be split up into two pieces

$$\mathcal{H} = \sum_{|n|\leq n_F}\sum_{\sigma} \hat{b}_{\sigma,n}\hat{b}^\dagger_{\sigma,n} + \sum_{|n|>n_F}\sum_{\sigma} \hat{a}^\dagger_{\sigma,n}\hat{a}_{\sigma,n}$$

Using the anti-commutation relations $\{\hat{b}_{\sigma,n}, \hat{b}^\dagger_{\sigma',m}\} = \delta_{\sigma,\sigma'}\delta_{nm}$, this can be brought to the form

$$\mathcal{H} = \sum_{|n|>n_F}\sum_{\sigma} \hat{a}^\dagger_{\sigma,n}\hat{a}_{\sigma,n} - \sum_{|n|\leq n_F}\sum_{\sigma} \hat{b}^\dagger_{\sigma,n}\hat{b}_{\sigma,n} + E_{gnd} \tag{43}$$

where

$$E_{gnd} = \sum_{\sigma}\sum_{|n|\leq n_F} E_n$$

Here, the excited states are given by the hole creation and electron creation operators acting on the filled vacuum, $|gnd\rangle$.

$$|n,\sigma\rangle_e = \hat{a}^\dagger_{n,\sigma}|gnd\rangle \quad \text{for} \quad |n| > n_F \qquad \text{and} \qquad |n,\sigma\rangle_h = \hat{b}^\dagger_{n,\sigma}|gnd\rangle \quad \text{for} \quad |n| < n_F \tag{44}$$

with excitation energies $E_{e,n} = E_n$ for $n > n_F$ and $E_{h,n} = -E_n$ for $n \leq n_F$ respectively.

6. The excited state with two particles, two holes and one particle and one hole are given by

$$\begin{gathered} |n,\sigma;n',\sigma'\rangle_e = \hat{a}^\dagger_{n,\sigma}\hat{a}^\dagger_{n',\sigma'}|gnd\rangle \qquad |n,\sigma;n',\sigma'\rangle_h = \hat{b}^\dagger_{n,\sigma}\hat{b}^\dagger_{n',\sigma'}|gnd\rangle \\ |n,\sigma\rangle_e \otimes |n',\sigma'\rangle_h = \hat{a}^\dagger_{n,\sigma}\hat{b}^\dagger_{n',\sigma'}|gnd\rangle \end{gathered} \tag{45}$$

After a brief exercise in manipulating anti-commutation relations, its straight forward to see that

$$\begin{gathered} \mathcal{H}|n,\sigma;n',\sigma'\rangle_e = (E_n + E_{n'})|n,\sigma;n',\sigma'\rangle_e \qquad \mathcal{H}|n,\sigma;n',\sigma'\rangle_h = -(E_n + E_{n'})|n,\sigma;n',\sigma'\rangle_h \\ \mathcal{H}|n,\sigma\rangle_e \otimes |n',\sigma'\rangle_h = (E_n - E_{n'})|n,\sigma\rangle_e \otimes |n',\sigma'\rangle_h \end{gathered} \tag{46}$$

To find the total number of fermions, we act on each state with the number operator

$$:\hat{N}:= \sum_{|n|>n_F} \sum_{\sigma} \hat{a}^{\dagger}_{n,\sigma}\hat{a}_{n,\sigma} - \sum_{|n|\leq n_F} \sum_{\sigma} \hat{b}^{\dagger}_{n,\sigma}\hat{b}_{n,\sigma}$$

Again, its straight forward to see that

$$:\hat{N}: \left|n,\sigma;n',\sigma'\right\rangle_e = 2\left|n,\sigma;n',\sigma'\right\rangle_e \qquad :\hat{N}: \left|n,\sigma;n',\sigma'\right\rangle_h = -2\left|n,\sigma;n',\sigma'\right\rangle_h$$
$$:\hat{N}: \left|n,\sigma\right\rangle_e \otimes \left|n',\sigma'\right\rangle_h = 0 \tag{47}$$

5 Thermodynamics of the Ideal Fermi Gas

Consider here an ideal spinless non-relativistic Fermi has at temperature T and density $1/v$ where v is the specific volume.

1. The grand canonical potential can be written as

$$\Xi = -k_B T V \int \frac{d^3p}{(2\pi\hbar)^3} \log\left(1 + e^{-\beta(\frac{p^2}{2m}-\mu)}\right)$$

and the number density is given by

$$\langle\rho\rangle = -\frac{1}{V}\frac{\partial\Xi}{\partial\mu} = \int \frac{d^3p}{(2\pi\hbar)^3} \frac{e^{-\beta(\frac{p^2}{2m}-\mu)}}{1+e^{-\beta(\frac{p^2}{2m}-\mu)}} = \frac{1}{2\pi^2\hbar^3}\int dp\, p^2 \frac{e^{-\beta(\frac{p^2}{2m}-\mu)}}{1+e^{-\beta(\frac{p^2}{2m}-\mu)}}$$

Defining $x = \beta p^2/2m$ and $z = e^{\beta\mu}$, one obtains

$$\langle\rho\rangle = \left(\frac{2\pi\beta\hbar^2}{m}\right)^{-3/2} \left[\frac{2}{\sqrt{\pi}}\int_0^\infty dx\,\sqrt{x}\frac{ze^{-x}}{1+ze^{-x}}\right] = \frac{1}{\lambda_T^3} f_{3/2}(z) \tag{48}$$

Similarly, the pressure can be found

$$P = -\frac{\partial\Xi}{\partial V} = k_B T \int \frac{d^3p}{(2\pi\hbar)^3} \log\left(1 + e^{-\beta(\frac{p^2}{2m}-\mu)}\right) = k_B T \left(\frac{2\pi\beta\hbar^2}{m}\right)^{-3/2} \left[\frac{2}{\sqrt{\pi}}\int_0^\infty dx\,\sqrt{x}\log(1+ze^{-x})\right]$$

This leads directly to the desired result.

$$\frac{P}{k_B T} = \frac{1}{\lambda_T^3} f_{5/2}(z) \tag{49}$$

2. When the density is low and *high* temperature, the equation of state can be approximated by a series in the density

$$\frac{P}{k_B T} = \rho + B\rho^2 + C\rho^3 + \ldots \tag{50}$$

where B is called the second virial coefficient, C the third virial coefficient, etc. To find the second virial coefficient, consider

$$\frac{P}{k_B T\rho} = 1 + B\rho = \frac{\sum_{n=1}^{\infty}(-1)^{n+1}z^n/n^{5/2}}{\sum_{n=1}^{\infty}(-1)^{n+1}z^n/n^{3/2}} \sim= 1 + \left(\frac{1}{2^{3/2}} - \frac{1}{2^{5/2}}\right) z + \ldots$$

In addition, $\rho \sim \frac{1}{\lambda_T^3}(z - \frac{z^2}{2^{3/2}})$. Hence to leading order in the fugacity $z = e^{\beta\mu}$ small,

$$\frac{P}{k_B T \rho} = 1 + \left(\frac{1}{2^{3/2}} - \frac{1}{2^{5/2}}\right) z + \ldots = 1 + \frac{B}{\lambda_T^3} z + \ldots \tag{51}$$

One can read off that $B \simeq .177\, \lambda_T^3$.

3. With a change of variables $\epsilon = \frac{p^2}{2m}$ its possible to rewrite the expression for ρ to give the desired form.

$$\rho = \frac{1}{4\pi^2}\left(\frac{2m}{\hbar^2}\right)^{3/2} \int_0^\infty d\epsilon \; \frac{\sqrt{\epsilon}}{e^{\beta(\epsilon-\mu)} + 1} \tag{52}$$

Now, the energy density is given by

$$u = \frac{U}{V} = \frac{1}{V}\left(\frac{1}{\beta}\frac{\partial \Xi}{\partial \beta} + \mu\right) = \int \frac{d^3 p}{(2\pi\hbar)^3} \; \frac{p^2}{2m} \frac{1}{e^{\beta(\frac{p^2}{2m} - \mu)} + 1} = \frac{1}{4\pi^2}\left(\frac{2m}{\hbar^2}\right)^{3/2} \int_0^\infty d\epsilon \; \frac{\epsilon^{3/2}}{e^{\beta(\epsilon-\mu)} + 1} \tag{53}$$

In going to the last equality, the substitution $\epsilon = \frac{p^2}{2m}$ as made.

4. Here we wish to study the specific heat of the system at *low temperatures.* This derivative with respect of temperature (at constant volume) of the energy density (53). The integral is of the form

$$\begin{aligned}
\int_0^\infty d\epsilon \; \frac{g(\epsilon)}{e^{\beta(\epsilon-\mu)} + 1} &= \int_0^\mu d\epsilon \; \frac{g(\epsilon)}{e^{\beta(\epsilon-\mu)} + 1} + \int_\mu^\infty d\epsilon \; \frac{g(\epsilon)}{e^{\beta(\epsilon-\mu)} + 1} \\
&= \int_0^\mu d\epsilon \; \frac{g(\epsilon) e^{-\beta(\epsilon-\mu)}}{e^{-\beta(\epsilon-\mu)} + 1} + \int_\mu^\infty d\epsilon \; \frac{g(\epsilon)}{e^{\beta(\epsilon-\mu)} + 1} \\
&= \int_0^\mu d\epsilon \; \frac{g(\epsilon) e^{-\beta(\epsilon-\mu)} + g(\epsilon) - g(\epsilon)}{e^{-\beta(\epsilon-\mu)} + 1} + \int_\mu^\infty d\epsilon \; \frac{g(\epsilon)}{e^{\beta(\epsilon-\mu)} + 1} \\
&= \int_0^\mu d\epsilon \; g(\epsilon) - \int_0^\mu d\epsilon \; \frac{g(\epsilon)}{e^{-\beta(\epsilon-\mu)} + 1} + \int_\mu^\infty d\epsilon \; \frac{g(\epsilon)}{e^{\beta(\epsilon-\mu)} + 1}
\end{aligned}$$

In the last integral, make the substitution $x = \beta(\epsilon - \mu)$ with $d\epsilon = dx/\beta$ and in the second $x = -\beta(\epsilon - \mu)$ with $d\epsilon = -dx/\beta$.

$$\int_0^\infty d\epsilon \; \frac{g(\epsilon)}{e^{\beta(\epsilon-\mu)} + 1} = \int_0^\mu d\epsilon \; g(\epsilon) - \int_0^{\beta\mu} \frac{dx}{\beta} \; \frac{g(\mu - x/\beta)}{e^x + 1} + \int_0^\infty \frac{dx}{\beta} \; \frac{g(\mu + x/\beta)}{e^x + 1}$$

Now comes the approximation. At low temperatures $\beta \to \infty$ and the limit of the second integral can be taken to infinity.

$$\int_0^\infty d\epsilon \; \frac{g(\epsilon)}{e^{\beta(\epsilon-\mu)} + 1} \sim \int_0^\mu d\epsilon \; g(\epsilon) + \int_0^\infty \frac{dx}{\beta} \; \frac{g(\mu + x/\beta)}{e^x + 1} - \int_0^\infty \frac{dx}{\beta} \; \frac{g(\mu - x/\beta)}{e^x + 1}$$

If $g(\mu \pm x/\beta)$ is a smooth analytic function of its argument and $x/\beta \to 0$ as $\beta \to \infty$, then it can be approximated by a series.

$$g(\mu \pm x/\beta) \sim g(\mu) \pm \frac{x}{\beta} g'(\mu)$$

So, the integral can be written as

$$\int_0^\infty d\epsilon \; \frac{g(\epsilon)}{e^{\beta(\epsilon-\mu)} + 1} \sim \int_0^\mu d\epsilon \; g(\epsilon) + \frac{2}{\beta^2} g'(\mu) \int_0^\infty \frac{dx}{\beta} \; \frac{x}{e^x + 1}$$

The remaining integral has value $\frac{\pi^2}{12}$. Its related to the Riemann ζ-function via its relationship to the Bernoulli numbers. The result is

$$\int_0^\infty d\epsilon \; \frac{g(\epsilon)}{e^{\beta(\epsilon-\mu)} + 1} \sim \int_0^\mu d\epsilon \; g(\epsilon) + \frac{\pi^2}{6\beta^2} g'(\mu)$$

Now, back to the problem at hand, $g(\epsilon) = \epsilon^{3/2}$. Identifying $\mu = \epsilon_F$ one finds

$$u \sim \frac{1}{4\pi^2}\left(\frac{2m}{\hbar^2}\right)^{3/2}\left(\int_0^{\epsilon_F} d\epsilon\ \epsilon^{3/2} + \frac{\pi^2}{6\beta^2}\frac{3}{2}\sqrt{\epsilon_F}\right) = \frac{1}{4\pi^2}\left(\frac{2m}{\hbar^2}\right)^{3/2}\left(\frac{5\epsilon^{5/2}}{2} + \frac{\pi^2}{4}\sqrt{\epsilon_F}k_B^2T^2\right)$$

So that the specific heat at constant volume is

$$C_V = \frac{\partial u}{\partial T} = \frac{1}{8}k_B^2\left(\frac{2m\epsilon_F^{1/3}}{\hbar^2}\right)^{3/2} T \tag{54}$$

A similar thing can be done for the density.

$$\rho = \frac{1}{4\pi^2}\left(\frac{2m}{\hbar^2}\right)^{3/2}\int_0^\infty d\epsilon\ \frac{\sqrt{\epsilon}}{e^{\beta(\epsilon-\mu)}+1} \sim \frac{1}{4\pi^2}\left(\frac{2m}{\hbar^2}\right)^{3/2}\left(\frac{2}{3}\epsilon_F^{3/2} + \frac{\pi^2}{12\beta^2}\frac{1}{\sqrt{\epsilon_F}}\right)$$

Near $T \sim 0$, the second term is much smaller than the first and can be neglected. This then gives a relationship between the density and the Fermi energy

$$\epsilon_F = \frac{\hbar^2}{2m}\left(\frac{6\pi^2}{v}\right)^{2/3} \tag{55}$$

The specific heat is then related to the density by

$$C_v = \frac{mk_B^2}{4\hbar^2}\left(\frac{6\pi^2}{v}\right)^{1/3} T \tag{56}$$

5. The pressure was found before in part 1:

$$\frac{P}{k_BT} = \frac{1}{\lambda_T^3}\left[\frac{2}{\sqrt{\pi}}\int_0^\infty dx\,\sqrt{x}\log(1+ze^{-x})\right]$$

Integrating by parts once, this can be re-written as

$$\frac{P}{k_BT} = \frac{1}{\lambda_T^3}\left[\frac{4}{3\sqrt{\pi}}\int_0^\infty dx\,x^{3/2}\frac{1}{z^{-1}e^x+1}\right] = \frac{1}{\lambda_T^3}\left[\frac{4}{3\sqrt{\pi}}\beta^{5/2}\int_0^\infty d\epsilon\,\epsilon^{3/2}\frac{1}{e^{\beta(\epsilon-\mu)}+1}\right]$$

with the substitution $x = \beta\epsilon$ and $z = e^{\beta\mu}$. This is of the form of the Sommerfeld expansion and the integral is approximately,

$$\frac{P}{k_BT} \sim \frac{1}{\lambda_T^3}\frac{4}{3\sqrt{\pi}}\beta^{5/2}\left[\int_0^{\epsilon_F} d\epsilon\,\epsilon^{3/2} + \frac{\pi^2}{6\beta^2}\frac{3}{2}\sqrt{\epsilon_F}\right] = \frac{1}{\lambda_T^3}\frac{4}{3\sqrt{\pi}}\beta^{5/2}\left[\frac{2}{5}\epsilon_F^{5/2} + \frac{\pi^2}{4\beta^2}\sqrt{\epsilon_F}\right]$$

Hence the limiting value of P as $T \to 0$ is

$$P_0 = \left(\frac{m}{2\pi\hbar^2\beta}\right)^{3/2}\frac{4}{3\sqrt{\pi}}\beta^{3/2}\left[\frac{2}{5}\epsilon_F^{5/2} + \frac{\pi^2}{4\beta^2}\sqrt{\epsilon_F}\right] = \frac{\rho}{\epsilon_F^{3/2}}\left[\frac{2}{5}\epsilon_F^{5/2} + \frac{\pi^2}{4}\sqrt{\epsilon_F}k_B^2T^2\right]$$

$$P_0 = \frac{2}{5}\rho\epsilon_F \tag{57}$$

One can interpret this non-zero pressure as coming from the Pauli exclusion principle.